The Symmetric Group

Table of contents

The definition . . . 1
The six elements of S_3 . . . 2
Composing . . . 2
Why it is a group . . . 4
The Cayley table of S_3 . . . 5
Powers and cyclic subgroups of S_3 . . . 5
Common mistakes . . . 7
See also . . . 7

S_n is the group whose elements are the bijections from $\{1, 2, \dots, n\}$ to itself, and whose operation is composition. Its elements are functions, not numbers, and it is the course's first group in which the order of a product matters. The definition of a group is in groups[1], and the two ingredients are one-to-one-and-onto-functions[2] and composition-of-functions[3]. First met in 2026-09-24-definitions-and-examples-of-groups[4].

The definition

> S_n is the group consisting of the set of all bijections from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, n\}$ together with the binary operation $\circ$ denoting composition of functions.

An element of S_n is a way of sending each of $1, \dots, n$ to one of $1, \dots, n$, so that no two go to the same place and every place is used. To combine two elements, do one and then the other.

How many elements. A bijection in S_3 can send 1 to any of three places. That leaves two places for 2, and then one for 3, so S_3 has $3 \cdot 2 \cdot 1 = 6$ elements. In general S_n has $n!$ elements, one for each arrangement of $1, \dots, n$ (counting-principles[5]).

[1] courses/math1061/groups.pdf
[2] courses/math1061/one-to-one-and-onto-functions.pdf
[3] courses/math1061/composition-of-functions.pdf
[4] courses/math1061/2026-09-24-definitions-and-examples-of-groups.pdf
[5] courses/math1061/counting-principles.pdf

The six elements of S_3

They are named f_1 to f_6 here, and each can be drawn as an arrow diagram (2026-09-24-definitions-and-examples-of-groups[6]). Each can also be written as a set of pairs or as a list of values.

	As a set of pairs	$1 \mapsto$	$2 \mapsto$	$3 \mapsto$	What it does
f_1	$\{(1,1),(2,2),(3,3)\}$	1	2	3	nothing
f_2	$\{(1,2),(2,1),(3,3)\}$	2	1	3	swaps 1 and 2
f_3	$\{(1,3),(2,2),(3,1)\}$	3	2	1	swaps 1 and 3
f_4	$\{(1,1),(2,3),(3,2)\}$	1	3	2	swaps 2 and 3
f_5	$\{(1,2),(2,3),(3,1)\}$	2	3	1	moves all three
f_6	$\{(1,3),(2,1),(3,2)\}$	3	1	2	moves all three, the other way

The pair form is the definition of a function as a set of pairs (functions-on-general-sets[7]). The names $f_1, \ldots, f_6$ are just labels, and other sources number the six differently.

Composing

$$(f \circ g)(x) = f(g(x))$$

The function on the right acts first. To work out $f \circ g$, take each of 1, 2, 3 in turn, apply g, then apply f to the result.

$f_2 \circ f_3$: f_3 first, then f_2.

x	$f_3(x)$	$f_2(f_3(x))$
1	3	3
2	2	1
3	1	2

[6] courses/math1061/2026-09-24-definitions-and-examples-of-groups.pdf
[7] courses/math1061/functions-on-general-sets.pdf

So $f_2 \circ f_3$ sends $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$, which is f_6.

$f_3 \circ f_2$: f_2 first, then f_3.

x	$f_2(x)$	$f_3(f_2(x))$
1	2	2
2	1	3
3	3	1

So $f_3 \circ f_2$ sends $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$, which is f_5.

The same two functions in the two orders give different answers. $f_2 \circ f_3 = f_6$ but $f_3 \circ f_2 = f_5$.

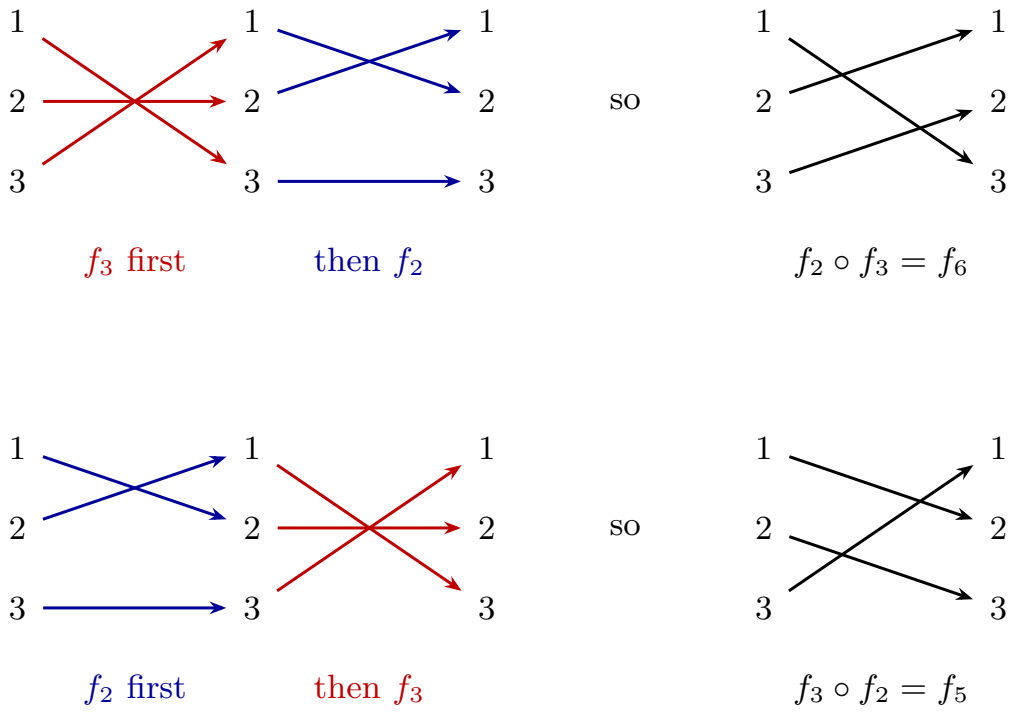


Figure 1: The same two elements of S_3 composed in both orders. Top: f_3 first and then f_2, which is $f_2 \circ f_3$. Following each number through both steps gives $1 \to 3$, $2 \to 1$, $3 \to 2$, the element f_6. Bottom: f_2 first and then f_3, which is $f_3 \circ f_2$. Now the result is $1 \to 2$, $2 \to 3$, $3 \to 1$, the element f_5. In each row the picture runs left to right in the order things happen, and the formula names the functions right to left.

Why it is a group

For S_3 the properties can be checked from its table, with associativity by a general argument (2026-09-24-definitions-and-examples-of-groups[8]). Each one also follows, for every n, from facts about functions:

- **Closure.** The composition of two bijections is a bijection (composition-of-functions[9]).
- **Associativity.** For any three functions, $f \circ (g \circ h)$ and $(f \circ g) \circ h$ both send x to $f(g(h(x)))$.
- **Identity.** The function that sends every number to itself, f_1 in S_3. Composing with it on either side changes nothing.
- **Inverses.** A bijection has an inverse function (inverse-functions[10]), which is again a bijection, and composing the two in either order gives the identity (composition-of-functions[11]).

So the inverse of f in the group S_n is the inverse function f^{-1}, and the two uses of the notation agree. In an arrow diagram the inverse is found by **reversing every arrow**.

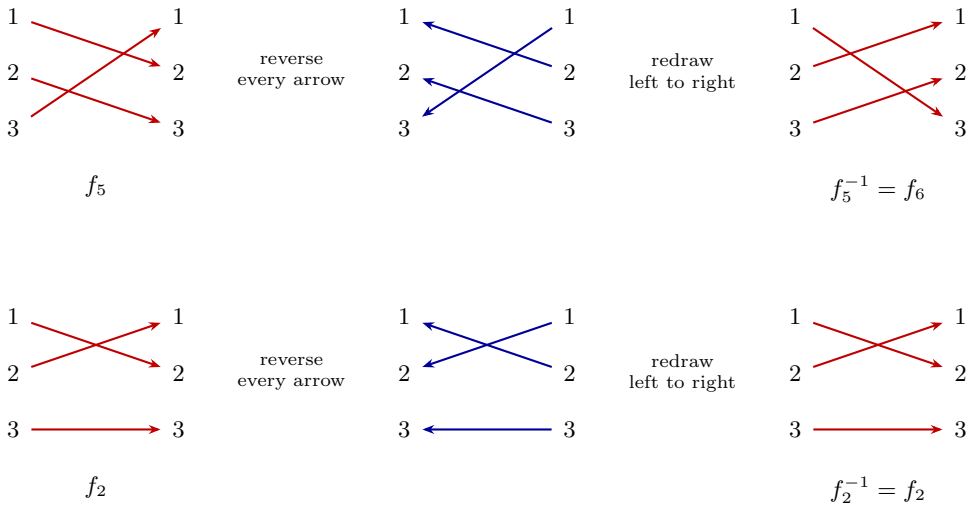


Figure 2: Inverses in S_3 by reversing arrows. Top: f_5 sends $1 \to 2$, $2 \to 3$, $3 \to 1$. Turning each arrow round gives $2 \to 1$, $3 \to 2$, $1 \to 3$, and reading that as a function of 1, 2, 3 gives $1 \to 3$, $2 \to 1$, $3 \to 2$, which is f_6. So $f_5^{-1} = f_6$. Bottom: f_2 swaps 1 and 2. Turning its arrows round gives the same diagram, so f_2 is its own inverse. The same holds for f_3 and f_4, the other two swaps.

[8] courses/math1061/2026-09-24-definitions-and-examples-of-groups.pdf
[9] courses/math1061/composition-of-functions.pdf
[10] courses/math1061/inverse-functions.pdf
[11] courses/math1061/composition-of-functions.pdf

The Cayley table of S_3

The entry in the row of f and the column of g is $f \circ g$ (cayley-tables[12]).

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
$\mathbf{f}_1$	f_1	f_2	f_3	f_4	f_5	f_6
$\mathbf{f}_2$	f_2	f_1	f_6	f_5	f_4	f_3
$\mathbf{f}_3$	f_3	f_5	f_1	f_6	f_2	f_4
$\mathbf{f}_4$	f_4	f_6	f_5	f_1	f_3	f_2
$\mathbf{f}_5$	f_5	f_3	f_4	f_2	f_6	f_1
$\mathbf{f}_6$	f_6	f_4	f_2	f_3	f_1	f_5

Read off it:

- **Identity:** f_1.
- **Inverses:** $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_4$, $f_5^{-1} = f_6$, $f_6^{-1} = f_5$.
- **Not abelian:** $f_2 \circ f_3 = f_6$ but $f_3 \circ f_2 = f_5$.

The three swaps are their own inverses: doing a swap twice puts everything back. f_5 and f_6 undo each other. The table also shows $f_5 \circ f_5 = f_6$ and $f_6 \circ f_5 = f_1$, so doing f_5 three times returns everything to where it started.

Non-abelian does not mean nothing commutes. $f_5 \circ f_6 = f_6 \circ f_5 = f_1$, and f_1 commutes with everything. A group is non-abelian as soon as **one** pair fails to commute.

Powers and cyclic subgroups of S_3

Powers are defined in any group (subgroups-and-cyclic-groups[13]). In S_3 a power is a repeated composition, $f^2 = f \circ f$, and $f^0 = f_1$. Read from the table:

- f_5**.** $f_5^1 = f_5$, $f_5^2 = f_5 \circ f_5 = f_6$, $f_5^3 = f_5^2 \circ f_5 = f_6 \circ f_5 = f_1$. This is "doing f_5 three times returns everything to where it started": f_5 moves $1 \to 2 \to 3 \to 1$, and three moves bring each number home. So $\langle f_5 \rangle = \{f_5, f_6, f_1\}$.
- f_6**.** $f_6 = f_5^{-1}$, so $\langle f_6 \rangle = \langle f_5 \rangle$. Directly: $f_6^2 = f_6 \circ f_6 = f_5$ and $f_6^3 = f_5 \circ f_6 = f_1$.
- **The swaps.** $f_2^2 = f_2 \circ f_2 = f_1$, since swapping twice changes nothing. So $\langle f_2 \rangle = \{f_2, f_1\}$, and likewise $\langle f_3 \rangle = \{f_3, f_1\}$ and $\langle f_4 \rangle = \{f_4, f_1\}$.
- **The identity.** $\langle f_1 \rangle = \{f_1\}$.

[12] courses/math1061/cayley-tables.pdf
[13] courses/math1061/subgroups-and-cyclic-groups.pdf

a	Powers of a until f_1	$\langle a \rangle$
f_1	f_1	$\{f_1\}$
f_2	f_2, f_1	$\{f_1, f_2\}$
f_3	f_3, f_1	$\{f_1, f_3\}$
f_4	f_4, f_1	$\{f_1, f_4\}$
f_5	f_5, f_6, f_1	$\{f_1, f_5, f_6\}$
f_6	f_6, f_5, f_1	$\{f_1, f_5, f_6\}$

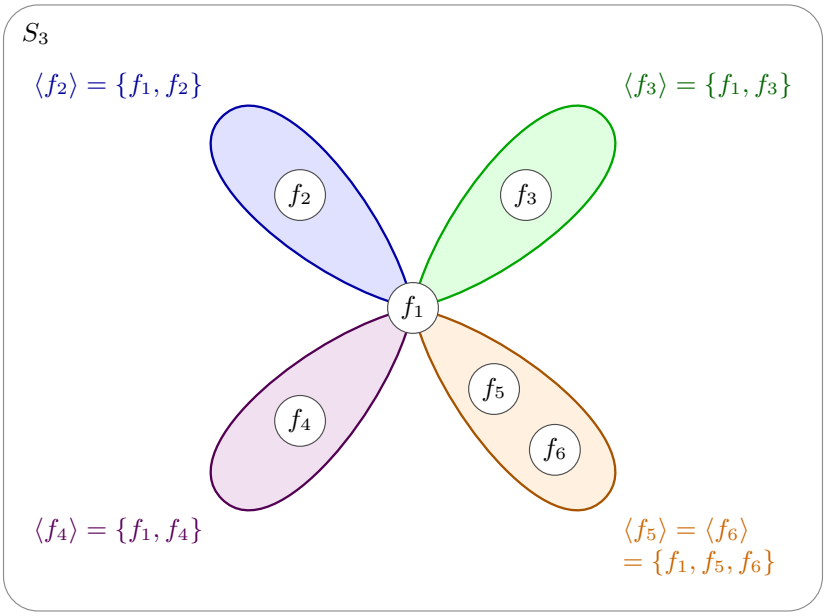


Figure 3: The cyclic subgroups of S_3, drawn inside it. Each petal is the set of powers of one element. The identity f_1, where the petals meet, is in every one, and it is the only element any two of them share. The three swaps f_2, f_3, f_4 each generate a subgroup of two elements, because a swap done twice changes nothing. f_5 and f_6 move all three numbers round and are each other's inverses, so they generate the same subgroup f_1, f_5, f_6. No petal is the whole of S_3, so no element generates it and S_3 is not cyclic.

None of the six is all of S_3, so S_3 **is not cyclic**. It could not have been: every cyclic group is abelian, and S_3 is not (subgroups-and-cyclic-groups[14]). $\{f_1, f_5, f_6\}$ also shows up as a block of the table when its elements are listed first (cayley-tables[15]).

[14] courses/math1061/subgroups-and-cyclic-groups.pdf
[15] courses/math1061/cayley-tables.pdf

Common mistakes

- **Composing in the wrong order.** $f \circ g$ means g first. Reading it left to right gives the other product, which in S_3 is usually a different element.
- **Reading the table as column $\circ$ row.**
- **Thinking S_3 has three elements.** It has six. The 3 is the size of the set being rearranged.
- **Treating the elements as numbers.** f_2 is a function. $f_2 \circ f_3$ is a composition, not a multiplication of 2 and 3.
- **Reading f^{-1} as $\frac{1}{f}$.** It is the inverse function.
- **Concluding from "non-abelian" that no two elements commute.**

See also

- groups[16] — the definition, and abelian against non-abelian
- cayley-tables[17] — reading a table, and why its rows and columns have no repeats
- composition-of-functions[18] — the operation of S_n, and why it is done right to left
- one-to-one-and-onto-functions[19] · inverse-functions[20] — bijections and their inverses, the elements of S_n and their group inverses
- functions-on-general-sets[21] — a function as a set of pairs
- subgroups-and-cyclic-groups[22] — powers, cyclic subgroups, and why a non-abelian group cannot be cyclic
- 2026-09-24-definitions-and-examples-of-groups[23] — Lecture 26 · 2026-10-06-elementary-properties-of-groups[24] — Lecture 27
- practice-problems[25] — §33, with full worked solutions

[16] courses/math1061/groups.pdf
[17] courses/math1061/cayley-tables.pdf
[18] courses/math1061/composition-of-functions.pdf
[19] courses/math1061/one-to-one-and-onto-functions.pdf
[20] courses/math1061/inverse-functions.pdf
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[25] courses/math1061/practice-problems.pdf