Reflexive, Symmetric and Transitive Relations

Table of contents

The definitions 1
Reading the definitions carefully 2
Proving and disproving 3
Checking a finite relation from its diagram 4
The examples side by side 5
Symmetric and transitive do not make reflexive 5
See also 6

The three properties a relation on a single set can have, what each looks like in an arrow diagram, and how to prove or disprove each one. Relations themselves are defined in relations-on-sets[1]. The lecture is 2026-09-17-relations-reflexivity-symmetry-transitivity[2].

The definitions

A relation ρ on a set A is

(1) reflexive if and only if $\forall x \in A,\ x\ \rho\ x$.

(2) symmetric if and only if $\forall x, y \in A,\ x\ \rho\ y \to y\ \rho\ x$.

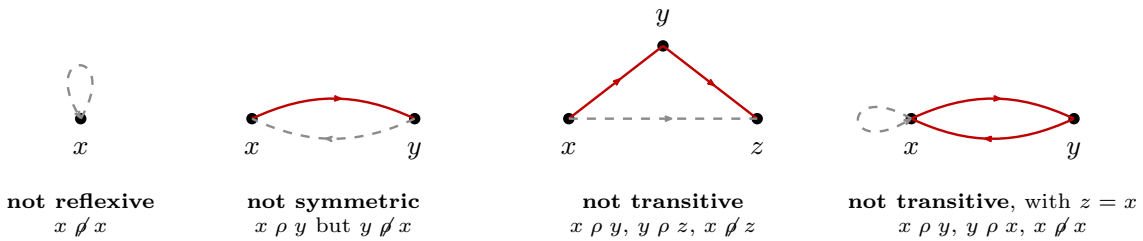
(3) transitive if and only if $\forall x, y, z \in A,\ (x\ \rho\ y \text{ and } y\ \rho\ z) \to x\ \rho\ z$.

In the arrow diagram:

- **reflexive** — every dot has a loop;
- **symmetric** — every arrow has a return arrow;
- **transitive** — every path of two arrows has a direct arrow from its start to its end, the **short cut**.

[1] courses/math1061/relations-on-sets.pdf
[2] courses/math1061/2026-09-17-relations-reflexivity-symmetry-transitivity.pdf

A relation fails a property when the diagram contains one of the patterns in Figure 1. Each is a **witness** — one specific element, pair or triple. Nothing more is needed to disprove a "for all".



not reflexive
$x \not\rho x$

not symmetric
$x \rho y$ but $y \not\rho x$

not transitive
$x \rho y$, $y \rho z$, $x \not\rho z$

not transitive, with $z = x$
$x \rho y$, $y \rho x$, $x \not\rho x$

Figure 1: What a failure of each property looks like; any one such pattern in a diagram disproves the property. Not reflexive: one element without a loop. Not symmetric: one arrow whose return is missing. Not transitive: one path of two arrows whose short cut is missing. The fourth panel is the not-transitive pattern with $z = x$: a round trip $x \to y \to x$ needs the short cut $x \to x$, a loop, so two elements joined both ways but without loops already break transitivity. The missing arrows are dashed.

Reading the definitions carefully

Reflexive is about every element. "$\forall x \in A$" means a single element without a loop is enough to make a relation not reflexive, even if every other element has one. Disjointness, $X \rho Y \leftrightarrow X \cap Y = \emptyset$ on $\mathcal{P}(\mathbb{Z}^+)$, is *not* reflexive, though $\emptyset \rho \emptyset$ holds.

Symmetric and transitive are conditional. They say what must happen **if** certain arrows are present. Where those arrows are absent, the condition holds automatically, by vacuous truth (conditional-statements[3]). This is why a relation with very few arrows can be symmetric and transitive. The empty relation on any set is both, and so is $\{(1,1)\}$ on $\{1,2\}$.

x, y, z need not be different. In the transitive condition, z may equal x. Then it reads: if $x \rho y$ and $y \rho x$, then $x \rho x$. **In a transitive relation, any two elements related both ways both have loops.** Three of the transitivity counterexamples in 2026-09-17-relations-reflexivity-symmetry-transitivity[4] use exactly this: $\{1\}$ and $\{2\}$ are disjoint both ways but $\{1\}$ is not disjoint from itself; in Diagram 1 $b \to a \to b$ exists but the loop at b does not; and for $xy \equiv 1 \pmod 7$, $2 \to 4 \to 2$ exists but $2 \not\rho 2$.

[3] courses/math1061/conditional-statements.pdf
[4] courses/math1061/2026-09-17-relations-reflexivity-symmetry-transitivity.pdf

Proving and disproving

Each property is a universal statement, so proving it needs a general argument about arbitrary elements, while disproving it needs one witness (quantified-statements[5], proof-techniques[6]):

	To prove it	To disprove it
reflexive	Let $x \in A$. Show $x \rho x$.	Find one $x \in A$ with $x \not\rho x$.
symmetric	Let $x, y \in A$ and suppose $x \rho y$. Show $y \rho x$.	Find $x, y \in A$ with $x \rho y$ but $y \not\rho x$.
transitive	Let $x, y, z \in A$ and suppose $x \rho y$ and $y \rho z$. Show $x \rho z$.	Find $x, y, z \in A$ with $x \rho y$ and $y \rho z$ but $x \not\rho z$.

In every proof, **unfold the definition of** ρ — turn $x \rho y$ into the condition it stands for, work with that, and fold it back up at the end. For a disproof, give specific elements and check each of the stated facts about them.

Worked: $m \rho n \leftrightarrow 6 \mid (m - n)$ on $\mathbb{Z}$

All three hold. Unfold with the definition of divides (divisibility-and-factorisation[7]): $m \rho n$ means $m - n = 6k$ for some $k \in \mathbb{Z}$.

- **Reflexive.** Let $m \in \mathbb{Z}$. Then $m - m = 0 = 6 \cdot 0$, so $6 \mid (m - m)$ and $m \rho m$.
- **Symmetric.** Let $m, n \in \mathbb{Z}$ with $m \rho n$, so $m - n = 6k$ for some $k \in \mathbb{Z}$. Then $n - m = 6(-k)$ and $-k \in \mathbb{Z}$, so $6 \mid (n - m)$ and $n \rho m$.
- **Transitive.** Let $m, n, p \in \mathbb{Z}$ with $m \rho n$ and $n \rho p$, so $m - n = 6k$ and $n - p = 6\ell$ for some $k, \ell \in \mathbb{Z}$. Adding, $m - p = 6(k + \ell)$ with $k + \ell \in \mathbb{Z}$, so $m \rho p$.

Its diagram on $\{0, \ldots, 11\}$ is drawn in relations-on-sets[8]. The second divisibility gets a **new variable** ℓ: reusing k would claim that $m - n$ and $n - p$ are the same multiple of 6.

Worked: $x \rho y \leftrightarrow x > y$ on $\mathbb{Z}$

- **Not reflexive.** $0 > 0$ is false, so $0 \not\rho 0$.
- **Not symmetric.** $1 \rho 0$ since $1 > 0$, but $0 \not\rho 1$ since $0 > 1$ is false.
- **Transitive.** Suppose $x > y$ and $y > z$. Then $x > z$, so $x \rho z$.

[5] courses/math1061/quantified-statements.pdf
[6] courses/math1061/proof-techniques.pdf
[7] courses/math1061/divisibility-and-factorisation.pdf
[8] courses/math1061/relations-on-sets.pdf

Worked: divides, on $\{1, 2, \ldots, 6\}$

$x \rho y \leftrightarrow x \mid y$ is reflexive ($x = x \cdot 1$), not symmetric ($1 \mid 2$ but $2 \nmid 1$), and transitive: if $y = xk$ and $z = y\ell$ then $z = x(k\ell)$. The diagram shows all three at once.

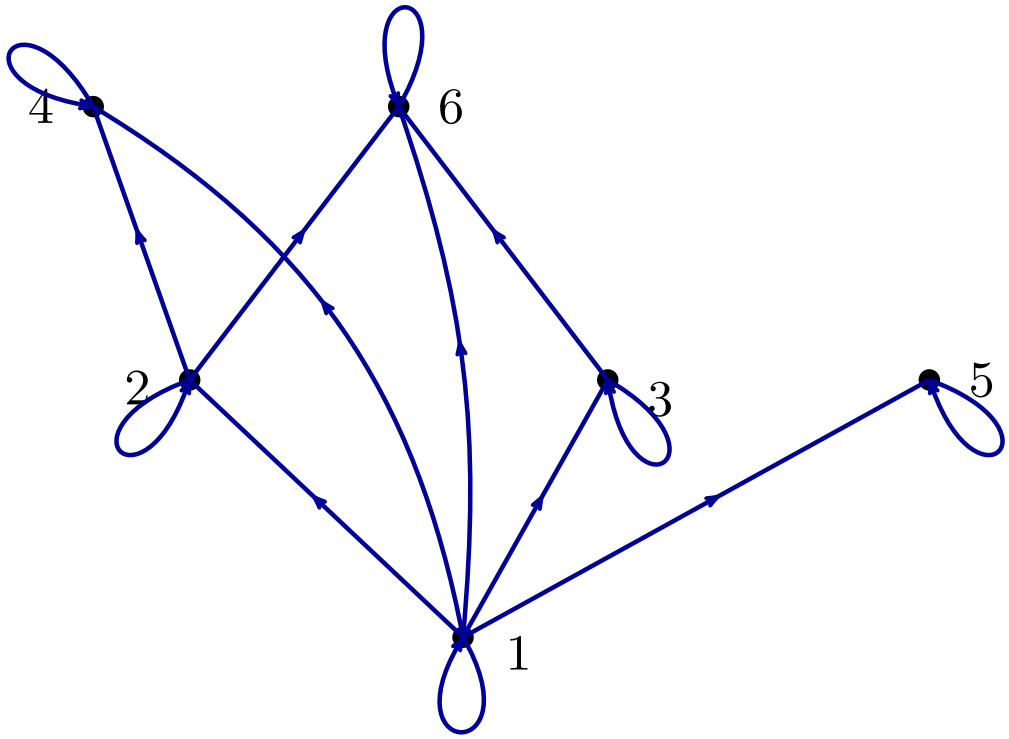


Figure 2: The relation $x \mid y$ on $1, 2, \ldots, 6$, with the numbers placed so every arrow points upward. Reflexive: a loop at every number, since each number divides itself. Not symmetric: no arrow comes back down — $1 \mid 2$, but $2 \nmid 1$. Transitive: every two-step path has its short cut, such as $1 \to 2 \to 4$ with the direct arrow $1 \to 4$. There are 14 arrows: 6 loops and 8 others.

Checking a finite relation from its diagram

For a relation given as a diagram or a list of pairs:

1. **Reflexive:** look for a loop at every element. Any element without one is the witness.

2. **Symmetric:** for each non-loop arrow, look for its return. Any arrow without one is the witness.
3. **Transitive:** for each arrow $x \to y$, look at every arrow leaving y, say $y \to z$, and check that $x \to z$ is there. Include the paths that come back, $x \to y \to x$, whose short cut is a loop. This is the slow check, and the round trips are where it usually goes wrong.

The same procedure answers "what must be added to make it reflexive, symmetric, transitive": the missing loops, the missing returns, and the missing short cuts. For transitivity, keep checking after adding, because a new arrow can create new two-step paths. This is worked on Diagram 1 in 2026-09-17-relations-reflexivity-symmetry-transitivity[9].

The examples side by side

Relation	on	Reflexive	Symmetric	Transitive
$X \subset Y$ (proper subset)	$\mathcal{P}(\{1,2,3\})$	no	no	**yes**
$X \cap Y = \emptyset$	$\mathcal{P}(\mathbb{Z}^+)$	no	**yes**	no
Diagram 1	$\{a,b,c,d\}$	no	no	no
$xy \equiv 1 \pmod 7$	$\{0,\dots,6\}$	no	**yes**	no
$\{(1,1)\}$	$\{1,2\}$	no	**yes**	**yes**
$6 \mid (m-n)$	$\mathbb{Z}$	**yes**	**yes**	**yes**
$x > y$	$\mathbb{Z}$	no	no	**yes**
$x \mid y$	$\{1,\dots,6\}$	**yes**	no	**yes**

The first five are worked in 2026-09-17-relations-reflexivity-symmetry-transitivity[10]; the other three are worked above. The three properties are independent, and most of the eight combinations appear here. Relations with all three, like $6 \mid (m-n)$, are **equivalence relations** (equivalence-relations[11]). Relations like $x \mid y$ — reflexive and transitive, with no two different elements related both ways — are **partial orders** (partial-order-relations[12]). Those are the next two topics in the course.

Symmetric and transitive do not make reflexive

> **Claim.** If a relation ρ on a set A is both symmetric and transitive, then ρ is reflexive. — **False.**

[9] courses/math1061/2026-09-17-relations-reflexivity-symmetry-transitivity.pdf
[10] courses/math1061/2026-09-17-relations-reflexivity-symmetry-transitivity.pdf
[11] courses/math1061/equivalence-relations.pdf
[12] courses/math1061/partial-order-relations.pdf

Counterexample. $A = \{1, 2\}$, $\rho = \{(1, 1)\}$. Symmetric: the only pair is $(1, 1)$, whose reverse is itself. Transitive: the only two-step path is $1 \to 1 \to 1$, whose short cut $(1, 1)$ is present. Not reflexive: $2 \not\rho\, 2$.

The tempting "proof" goes: *let $x \in A$; take y with $x \rho y$; by symmetry $y \rho x$; by transitivity $x \rho x$.* Every step is valid except the second. **Nothing guarantees there is any y with $x \rho y$.** The element 2 in the counterexample is related to nothing, so there is nothing to start the argument from. Adding the missing hypothesis makes the claim true:

> If ρ is symmetric and transitive, **and for each $x \in A$ there is some $y \in A$ with $x \rho y$**, then ρ is reflexive.

The proof is the tempting argument above, now legitimate. The picture is the right-hand panel of 2026-09-17-relations-reflexivity-symmetry-transitivity[13].

See also

- relations-on-sets[14] — relations as sets of pairs, the notation, arrow diagrams
- equivalence-relations[15] · partial-order-relations[16] — the two kinds of relation built from these properties, the second with antisymmetry[17] in place of symmetry
- quantified-statements[18] · proof-techniques[19] — why a "for all" needs a general argument and its negation needs a witness
- conditional-statements[20] — vacuous truth, which is why sparse relations are often symmetric and transitive
- sets-and-set-notation[21] — the proper-subset relation, and why it has no loops
- 2026-09-17-relations-reflexivity-symmetry-transitivity[22] — Lecture 23
- practice-problems[23] — §30, with full worked solutions

[13] courses/math1061/2026-09-17-relations-reflexivity-symmetry-transitivity.pdf
[14] courses/math1061/relations-on-sets.pdf
[15] courses/math1061/equivalence-relations.pdf
[16] courses/math1061/partial-order-relations.pdf
[17] courses/math1061/antisymmetric-relations.pdf
[18] courses/math1061/quantified-statements.pdf
[19] courses/math1061/proof-techniques.pdf
[20] courses/math1061/conditional-statements.pdf
[21] courses/math1061/sets-and-set-notation.pdf
[22] courses/math1061/2026-09-17-relations-reflexivity-symmetry-transitivity.pdf
[23] courses/math1061/practice-problems.pdf