Rational & Irrational Numbers

Table of contents

The definitions and closure results that make $\mathbb{Q}$ a workable object in proofs, plus the classic irrationality argument.

Definitions

$$r \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z} \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0$$

$$r \in \overline{\mathbb{Q}} \iff r \notin \mathbb{Q}$$

$\overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{Q}$ is the set of **irrational** numbers. Examples: $\frac{3}{4} \in \mathbb{Q}$; $\sqrt{2} \in \overline{\mathbb{Q}}$; $\pi \in \overline{\mathbb{Q}}$.

Note the definition is existential: to show r is rational you must **exhibit** integers a and b with $b \neq 0$. Nearly every proof below ends by checking three things — the numerator is an integer, the denominator is an integer, and the denominator is non-zero.

Fact: every rational number can be written as a fraction in **lowest terms** (numerator and denominator have no common factors).

Why irrationality statements want contradiction or contraposition

"r is irrational" is a *negative* statement — $r \notin \mathbb{Q}$ — so a direct proof has nothing to unfold. Assuming the negation instead ($r \in \mathbb{Q}$) hands you $r = \frac{a}{b}$ to work with. See proof-techniques[1] for the general shape.

Concretely, for **the sum of any rational number and any irrational number is irrational**:

- Symbolically: $\forall r, s \in \mathbb{R}$, if $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ then $r + s \in \overline{\mathbb{Q}}$ — or equivalently $\forall r \in \mathbb{Q}$ and $\forall s \in \mathbb{R}$, if $s \in \overline{\mathbb{Q}}$ then $r + s \in \overline{\mathbb{Q}}$.
- **Direct proof**: suppose $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ — and now you're stuck, because $s \in \overline{\mathbb{Q}}$ gives you nothing to substitute.
- **Contradiction**: suppose $\exists r \in \mathbb{Q}, s \in \mathbb{R}$ with $s \in \overline{\mathbb{Q}}$ and $r + s \in \mathbb{Q}$.
- **Contraposition**: prove $\forall r \in \mathbb{Q}$ and $\forall s \in \mathbb{R}$, if $r + s \in \mathbb{Q}$ then $s \in \mathbb{Q}$.

  Proof by contraposition. Suppose $r \in \mathbb{Q}$, $s \in \mathbb{R}$ and $r + s \in \mathbb{Q}$. Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Since $r + s \in \mathbb{Q}$, $r + s = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Now

  $$s = (r + s) - r = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}.$$

  Since $a, b, c, d \in \mathbb{Z}$ we know $bc - ad \in \mathbb{Z}$ and $bd \in \mathbb{Z}$; and $bd \neq 0$ since $b \neq 0$ and $d \neq 0$. Thus $s \in \mathbb{Q}$. ☐

The trick that makes it work is writing the thing you want in terms of the things you have: $s = (r + s) - r$.

Closure results

$\forall r \in \mathbb{Q}, (2r^2 - 3r + 1) \in \mathbb{Q}$ — direct proof is best here; contraposition gets stuck.

> Let $r \in \mathbb{Q}$, so $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Then
>
> $$2r^2 - 3r + 1 = \frac{2a^2}{b^2} - \frac{3ab}{b^2} + \frac{b^2}{b^2} = \frac{2a^2 - 3ab + b^2}{b^2}.$$
>
> Since $a, b \in \mathbb{Z}$ we have $2a^2 - 3ab + b^2 \in \mathbb{Z}$ and $b^2 \in \mathbb{Z}$, and $b^2 \neq 0$ since $b \neq 0$. So $2r^2 - 3r + 1 \in \mathbb{Q}$. ☐

$\forall r \in \mathbb{R}$**, if** r^2 **is irrational then** r **is irrational.** Proved by contraposition in proof-techniques[2] — the contrapositive is "if $r \in \mathbb{Q}$ then $r^2 \in \mathbb{Q}$".

For real m, n**: if** m **is irrational and** n **is rational, then** $m - n$ **is irrational.**

[1] courses/math1061/proof-techniques.pdf
[2] courses/math1061/proof-techniques.pdf

Proof by contradiction. Suppose m is irrational, n is rational, but $m - n$ is rational. Then $n = \frac{a}{b}$ and $m-n = \frac{c}{d}$ for integers a, b, c, d with $b, d \neq 0$. Since $m = (m-n)+n$,

$$m = \frac{c}{d} + \frac{a}{b} = \frac{bc + ad}{bd},$$

with $bc+ad \in \mathbb{Z}$, $bd \in \mathbb{Z}$ and $bd \neq 0$. So m is rational, contradicting the assumption. $\square$

For a real number r: if $2r^2 - 3r$ is irrational, then r is irrational.

Proof by contraposition. The contrapositive is: if $r \in \mathbb{Q}$ then $2r^2 - 3r \in \mathbb{Q}$. Let $r = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$. Then $2r^2 - 3r = \frac{2a^2 - 3ab}{b^2}$, with integer numerator and non-zero integer denominator. $\square$

For all real r, s and n: if r is irrational, s is rational and n is a positive integer, then $n(r - s)$ is irrational.

Proof by contradiction. Suppose otherwise: $r \in \overline{\mathbb{Q}}$, $s \in \mathbb{Q}$, $n \in \mathbb{Z}^+$, but $n(r-s) \in \mathbb{Q}$. Write $s = \frac{a}{b}$ and $n(r-s) = \frac{c}{d}$ with $b, d \neq 0$. Since $n \neq 0$,

$$r = s + \frac{c}{d} \cdot \frac{1}{n} = \frac{a}{b} + \frac{c}{dn} = \frac{adn + bc}{bdn}.$$

Here $adn + bc \in \mathbb{Z}$, $bdn \in \mathbb{Z}$, and $bdn \neq 0$ since b, d, n are all non-zero. So $r \in \mathbb{Q}$ — a contradiction. $\square$

For every rational r, if $r \neq 0$ then $r\sqrt{2}$ is irrational. Negation: $\exists r \in \mathbb{Q}$ such that $r \neq 0$ and $r\sqrt{2} \in \mathbb{Q}$.

Proof by contradiction. Suppose the negation holds. Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and $a \neq 0$ because $r \neq 0$. Since $r\sqrt{2} \in \mathbb{Q}$, $r\sqrt{2} = \frac{c}{d}$ for integers c, d with $d \neq 0$. Then

$$\sqrt{2} = \frac{r\sqrt{2}}{r} = \frac{c}{d} \cdot \frac{b}{a} = \frac{bc}{ad},$$

where $bc \in \mathbb{Z}$, $ad \in \mathbb{Z}$ and $ad \neq 0$ (since $a \neq 0$ and $d \neq 0$). So $\sqrt{2} \in \mathbb{Q}$ — contradicting the theorem below. $\square$

$\sqrt{2}$ is irrational

The set-up needs two ingredients: the lowest-terms fact above, and the lemma **for all integers n, if n^2 is even then n is even** (proved by contraposition — see proof-techniques[3]).

[3] courses/math1061/proof-techniques.pdf

Proof by contradiction. Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and we may assume $\frac{a}{b}$ is in lowest terms.

Squaring, $2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$ and thus a^2 is even. By the lemma, a is even, so $a = 2\ell$ for some $\ell \in \mathbb{Z}$.

Substituting: $2b^2 = a^2 = (2\ell)^2 = 4\ell^2$, so $b^2 = 2\ell^2$ and b^2 is even. By the lemma again, b is even.

But now a and b are both even, so they share the common factor 2 — contradicting the assumption that $\frac{a}{b}$ is in lowest terms. Therefore $\sqrt{2} \in \overline{\mathbb{Q}}$. □

Neither a direct proof nor a contraposition proof gets off the ground here: the statement isn't a conditional with a usable hypothesis, so contradiction is the only one of the three that has anywhere to start.

See also

- proof-techniques[4] — direct / counterexample / contradiction / contraposition
- quantified-statements[5]
- divisibility-and-factorisation[6]
- practice-problems[7] — §12 Rational Numbers, with full worked solutions

[4] courses/math1061/proof-techniques.pdf
[5] courses/math1061/quantified-statements.pdf
[6] courses/math1061/divisibility-and-factorisation.pdf
[7] courses/math1061/practice-problems.pdf