Quantified Statements

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Where logical-connectives[1] and conditional-statements[2] deal with whole statements, **predicates** let us talk about statements whose truth depends on a variable, and **quantifiers** turn those predicates back into statements that are definitely true or false.

Predicates

A **predicate** is a sentence with a finite number of variables that becomes a statement when particular values are assigned to those variables.

- Predicates: $x > 0$, "x divides y".
- Not predicates: "Is it raining?" (a question), "She is happy." (no objective truth value).

A predicate is written $P(x)$, and the set of values x may take is its **domain** D.

Common domains

[1] courses/math1061/logical-connectives.pdf
[2] courses/math1061/conditional-statements.pdf

Symbol	Set
$\mathbb{R}$	real numbers
$\mathbb{Q}$	rational numbers
$\overline{\mathbb{Q}}$	irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$)
$\mathbb{Z}$	integers, $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{N}$	natural numbers
$\mathbb{Z}^+$	positive integers, $\{1, 2, 3, \dots\}$
$\mathbb{Z}^{\text{nonneg}} = \mathbb{Z}^{\geq 0}$	$\{0, 1, 2, 3, \dots\}$
$\mathbb{Z}^{\text{even}}$	$\{\dots, -4, -2, 0, 2, 4, \dots\}$

Nesting: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, with $\overline{\mathbb{Q}}$ sitting alongside $\mathbb{Q}$ inside $\mathbb{R}$.

Take care with $\mathbb{N}$ — different sources define it as $\{0, 1, 2, \dots\}$ *or* $\{1, 2, 3, \dots\}$. Prefer the unambiguous $\mathbb{Z}^+$ or $\mathbb{Z}^{\geq 0}$ in your own writing.

$\in$ means "belongs to" / "is an element of"; $\notin$ means "is not an element of". Note that $\{\mathbb{Z}\} \neq \mathbb{Z}$ — writing $x \in \{\mathbb{Z}\}$ says x *is* the set $\mathbb{Z}$, not that x is an integer.

The two quantifiers

Symbol	Reads	Associated English words
$\forall$	"for all"	all, every, any, each, none, no
$\exists$	"there exists"	there is, some, at least one, not all

- **Universal**: $\forall x \in D, P(x)$ — "for all x in D, $P(x)$".
- **Existential**: $\exists x \in D$ such that $P(x)$ — "there exists an x in D such that $P(x)$". "such that" is commonly abbreviated **s.t.**

Note the punctuation convention: $\forall$ takes a comma, $\exists$ takes "such that".

Translating English to symbols

English	Symbolic
Every rational number is a real number.	$\forall x \in \mathbb{Q}, x \in \mathbb{R}$
No integer is an irrational number.	$\forall x \in \mathbb{Z}, x \notin \overline{\mathbb{Q}}$
Some real numbers are rational.	$\exists x \in \mathbb{R}$ s.t. $x \in \mathbb{Q}$
Not all rational numbers are integers.	$\exists x \in \mathbb{Q}$ s.t. $x \notin \mathbb{Z}$
All prime numbers are odd.	$\forall x \in P, x$ is odd

English	Symbolic
If an integer is divisible by 2 then it is divisible by 4.	$\forall x \in \mathbb{Z}$, if $2 \mid x$ then $4 \mid x$

Two things to watch:

- **"No …" is universal, not existential** — "no integer is irrational" becomes $\forall$, with the predicate negated.
- **Sometimes the quantifier is implicit.** "If an integer is a square, then it is nonnegative" carries a hidden $\forall$: $\forall n \in \mathbb{Z}$, if n is a square then $n \geq 0$.

Determining truth

- $\forall x \in D, P(x)$ is shown **false** by one counterexample.
- $\exists x \in D$ s.t. $P(x)$ is shown **true** by one example.

The asymmetry matters: a single value can settle a universal statement only in the negative, and an existential statement only in the positive.

$\exists y \in \mathbb{Z}$ s.t. $y^2 - y = 0$ is **true** — choose $y = 0$ or $y = 1$. That some other choice ($y = 2$) fails is irrelevant; $\exists$ only asks for one witness. The corresponding universal statement $\forall y \in \mathbb{Z}, y^2 - y = 0$ is false, and $y = 2$ is exactly the counterexample that shows it.

$\forall x \in \mathbb{Z}$, if 6 is divisible by x then $x = 2$ is **false** — take $x = 3$: it's an integer, 6 is divisible by 3, but $3 \neq 2$. (The corrected true statement is $\forall x \in \mathbb{Z}$, if 6 is divisible by x then $x = \pm 1, \pm 2, \pm 3$ or ± 6.)

Negation

Negating a quantified statement **flips the quantifier and negates the predicate**:

$$\sim (\forall x \in D, P(x)) \equiv \exists x \in D \text{ s.t. } \sim P(x)$$

$$\sim (\exists x \in D \text{ s.t. } P(x)) \equiv \forall x \in D, \sim P(x)$$

If the negation of a statement is true, the original is false — so proving the negation is a legitimate way to disprove a statement (see proof-techniques[3]).

[3] courses/math1061/proof-techniques.pdf

When the predicate is itself conditional, negate it with $\sim (p \to q) \equiv p \wedge \sim q$ (see logical-equivalence-laws[4]). So the negation of

$$\forall x \in \mathbb{Z},\ \text{if } x^2 > 3 \text{ then } x > 3$$

is $\exists x \in \mathbb{Z}$ s.t. $x^2 > 3$ **and** $x \leq 3$ — *not* "if … then …" with an $\exists$ in front, and not the converse.

A negation is not just "the opposite-sounding statement"

For $S = \{1,3,5,7\}$, $\{1,2,3,4\}$, $\{2,4,6,8\}$:

S	$\forall x \in S, x$ is odd	$\forall x \in S, x$ is not odd	$\exists x \in S$ s.t. x is not odd
$\{1,3,5,7\}$	T	F	F
$\{1,2,3,4\}$	F	F	T
$\{2,4,6,8\}$	F	T	T

The first and *third* columns are exact negations of each other (they disagree in every row). The middle column — "all are not odd" — is **not** the negation: for $S = \{1,2,3,4\}$ both it and the original are false.

Worked negations

Statement	Negation	Which is true
$\exists y \in \mathbb{R}$ s.t. $y^2 < 0$	$\forall y \in \mathbb{R}, y^2 \geq 0$	negation
All dogs are white.	Some dog is not white.	negation
$\forall x \in \mathbb{Z}^+, x$ is prime or x is composite	$\exists x \in \mathbb{Z}^+$ s.t. x is neither prime nor composite	negation ($x = 1$)
$\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$	$\exists x \in \mathbb{R}$ s.t. $x > 2$ and $x^2 \leq 4$	original
$\forall x \in \mathbb{R}$, if $x^2 > 9$ then $x \geq 2$	$\exists x \in \mathbb{R}$ s.t. $x^2 > 9$ and $x < 2$	negation ($x = -4$)

[4] courses/math1061/logical-equivalence-laws.pdf

Multiple quantifiers

Order matters. Read the statement left to right as a game:

- $\forall x \in X, \exists y \in Y$ s.t. $P(x, y)$ — "someone gives me an x; can I *always* then find a y?" The choice of y may depend on x.
- $\exists x \in X$ s.t. $\forall y \in Y, P(x, y)$ — "is there a *single* x that works for *every* y?" The x must be chosen before seeing y.

Both of these happen to be true for $xy \geq y$ over $\mathbb{Z}$, but for different reasons:

1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $xy \geq y$ — given x, choose $y = x$ (since $x^2 \geq x$ for every integer), or more simply $y = 0$.
2. $\exists x \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{Z}, xy \geq y$ — take $x = 1$, since $y \geq y$ always.

Where order really bites: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $y = 2x$ is **true** (given x, take $y = 2x$), but $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x = 2y$ is **false** — take $x = 3$, and $y = \frac{3}{2} \notin \mathbb{Z}$.

Writing two of the same quantifier

$\exists x \in D$ and $\exists y \in E \ldots$ is the clearest form; $\exists x \in D$ and $y \in E \ldots$ is common shorthand, and $\exists x, y \in D \ldots$ is shorthand when both range over the same domain.

Negating multiple quantifiers

Flip **every** quantifier in order, and negate the predicate at the end:

$$\sim \big(\forall x \in X, \exists y \in Y \text{ s.t. } P(x, y)\big) \equiv \exists x \in X \text{ s.t. } \forall y \in Y, \sim P(x, y)$$

Worked examples:

Statement	Negation	Which is true
$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x + y = 0$	$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, x + y \neq 0$	original (take $y = -x$)
$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = 1$	$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, xy \neq 1$	negation ($x = 0$)
$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, xy = 0$	$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy \neq 0$	original (take $x = 0$)
$\exists x \in \mathbb{Z}^+$ s.t. $\forall y \in \mathbb{Z}^+, x \geq y$	$\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+$ s.t. $x < y$	negation (take $y = x + 1$)
$\forall x, y \in \mathbb{Z}, \exists z \in \mathbb{Z}$ s.t. $z = x - y$	$\exists x, y \in \mathbb{Z}$ s.t. $\forall z \in \mathbb{Z}, z \neq x - y$	original
$\forall r, s \in \mathbb{R}$, if $rs \in \mathbb{Q}$ then $r \in \mathbb{Q}$ and $s \in \mathbb{Q}$	$\exists r, s \in \mathbb{R}$ s.t. $rs \in \mathbb{Q}$ and ($r \notin \mathbb{Q}$ or $s \notin \mathbb{Q}$)	negation ($r = s = \sqrt{2}$)

A negation trap: the negation of "$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x + y$ is odd" is "**there exists** an integer x such that **for every** integer y, $x + y$ is even". (The original is true: if x is odd take $y = 0$, if x is even take $y = 1$.)

See also

- logical-connectives[5] · logical-equivalence-laws[6] · conditional-statements[7]
- proof-techniques[8] — proving and disproving quantified statements
- practice-problems[9] — §6 Quantified Statements, with full worked solutions
- practice-problems[10] — §7 Negation of Quantified Statements, with full worked solutions
- practice-problems[11] — §8 Statements with Multiple Quantifiers, with full worked solutions

[5] courses/math1061/logical-connectives.pdf
[6] courses/math1061/logical-equivalence-laws.pdf
[7] courses/math1061/conditional-statements.pdf
[8] courses/math1061/proof-techniques.pdf
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