One-to-One and Onto Functions

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The two properties a function $f : X \to Y$ can have beyond being a function at all — **injective (one-to-one)** and **surjective (onto)** — with the proof templates for showing each and for showing its failure. Together they make a **bijection**, which is what inverse-functions[1] needs. The definition of a function is in functions-on-general-sets[2]; the lecture is 2026-09-07-functions-one-to-one-onto-inverse[3].

The definitions

Let f be a function, $f : X \to Y$.

> **Injective function (one-to-one function):** $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
>
> **Surjective function (onto function):** $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.
>
> A function that is both one-to-one and onto is a **bijection**.

[1] courses/math1061/inverse-functions.pdf
[2] courses/math1061/functions-on-general-sets.pdf
[3] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf

In words: **one-to-one** means no two inputs share an output — different inputs, different outputs. **Onto** means every element of the co-domain is hit — the range is the whole of Y. Each name has a Latin and an English form, used interchangeably: *injective / one-to-one*, *surjective / onto*, *bijective / one-to-one correspondence*.

The two are about opposite ends of the arrows. One-to-one is a condition on the **domain side**: arrows never merge. Onto is a condition on the **co-domain side**: nothing is left unreached. They are independent, and all four combinations occur:

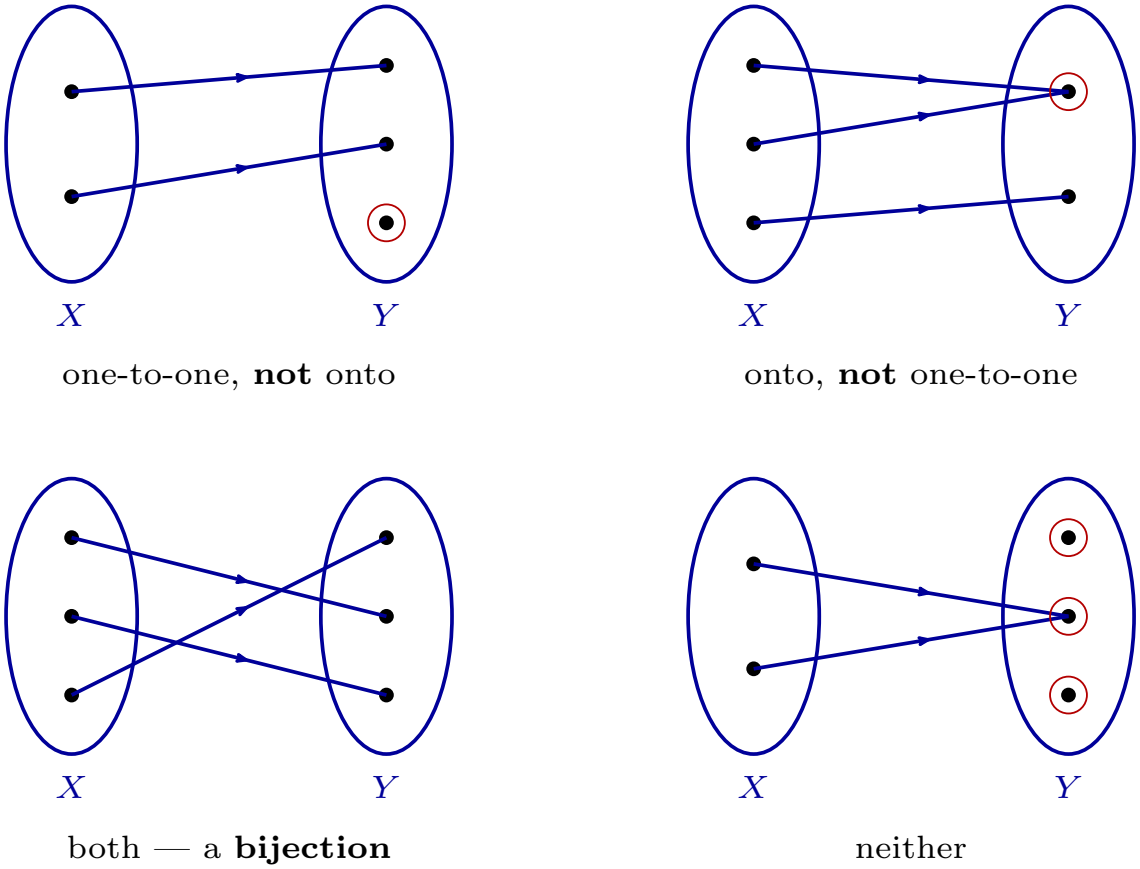


Figure 1: The four combinations, on the smallest sets that show them. Top left: one-to-one but not onto — arrows never merge, but the third dot of Y is missed. Top right: onto but not one-to-one — everything in Y is hit, but two arrows merge. Bottom left: both, a bijection — a perfect pairing. Bottom right: neither. A red ring marks the offending element of Y in each failing panel — a target that two arrows share, or a dot no arrow reaches; in the *neither* panel it marks one of each kind.

For finite sets the picture gives a counting fact for free: one-to-one forces $|X| \leq |Y|$, onto

forces $|X| \geq |Y|$, and a bijection forces $|X| = |Y|$. That is the beginning of the next topic, cardinality[4], where the same three facts become the definitions of $\leq$, $\geq$ and $=$ for sets of any size.

On the real line

The graph versions of the two tests:

	Test on the graph	Example
one-to-one	no horizontal line meets the graph twice	2^x passes; $x^3 - x$ fails, x^2 fails
onto $\mathbb{R}$	every horizontal line meets the graph at least once	$x^3 - x$ passes; 2^x fails (never ≤ 0), x^2 fails

The *vertical* line test decides whether a curve is a function at all (functions-on-general-sets[5]); the *horizontal* line test decides one-to-one.

Changing the co-domain

Onto depends on the **declared** co-domain, so it can be switched on by declaring a smaller one:

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 \text{ is not onto;} \qquad g : \mathbb{R} \to \mathbb{R}^{\geq 0},\ g(x) = x^2 \text{ is onto.}$$

Same pairs, different function, because Y is part of what a function is. Any function becomes onto its own range. One-to-one, by contrast, is a fact about the pairs alone and cannot be adjusted by relabelling the co-domain — though it *can* be switched on by shrinking the domain, which is how $\sqrt{\ }$ gets to be the inverse of x^2 on $\mathbb{R}^{\geq 0}$.

How to prove it

To show $f : X \to Y$ is …	Template
one-to-one	Suppose $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. … Hence $x_1 = x_2$. $\square$

[4] courses/math1061/cardinality.pdf
[5] courses/math1061/functions-on-general-sets.pdf

To show $f : X \to Y$ is …	Template
not one-to-one	Exhibit $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.
onto	Let $y \in Y$. Take $x =$ … *(in terms of y).* Check $x \in X$. Check $f(x) = y$. $\square$
not onto	Exhibit $y \in Y$ such that $f(x) \neq y$ for all $x \in X$ *(and say why).*

The shape follows the quantifiers (quantified-statements[6]). One-to-one and onto are both **universal** statements, so proving them means starting from an arbitrary element and arguing in general; their negations are **existential**, so disproving means producing one witness. This is the same logic as the element method for sets (proving-set-properties[7]), and the same warning applies: **checking examples proves neither property**, it can only disprove one.

One-to-one: algebra

$g : \mathbb{Z} \to \mathbb{Z}^{\text{odd}}$, $g(x) = 2x + 3$.

Let $x_1, x_2 \in \mathbb{Z}$ with $g(x_1) = g(x_2)$. Then $2x_1 + 3 = 2x_2 + 3$, so $2x_1 = 2x_2$, so $x_1 = x_2$. $\therefore g$ is one-to-one. $\square$

The proof is "undo the function": peel operations off both sides until the inputs are exposed. Every step must be reversible-in-effect — dividing by 2 is fine; squaring both sides would not be, which is exactly why x^2 is not one-to-one.

Onto: solve backwards, then run forwards

Same g. **Scratch work** (not part of the proof): want x with $2x + 3 = y$, so $x = \frac{y-3}{2}$.

Proof. Let $y \in \mathbb{Z}^{\text{odd}}$. Take $x = \frac{y-3}{2}$. Since y is odd, $y - 3$ is even, so $x \in \mathbb{Z}$. And $g(x) = 2 \cdot \frac{y-3}{2} + 3 = y$. $\therefore g$ is onto. $\square$

Three moves, always in this order: **name** the candidate, **check it is in the domain**, **check it maps to** y. The middle one is where onto proofs go wrong. Here it is the only place the oddness of y is used; drop it and the same lines "prove" that $2x + 3$ is onto all of $\mathbb{Z}$, which is false ($y = 4$ needs $x = \frac{1}{2}$).

[6] courses/math1061/quantified-statements.pdf
[7] courses/math1061/proving-set-properties.pdf

Pairs as inputs

$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $f((x, y)) = (y + 1, 2x)$.

One-to-one. Suppose $f((x_1, y_1)) = f((x_2, y_2))$. Then $(y_1 + 1, 2x_1) = (y_2 + 1, 2x_2)$, so $y_1 + 1 = y_2 + 1$ **and** $2x_1 = 2x_2$; hence $y_1 = y_2$ **and** $x_1 = x_2$, so $(x_1, y_1) = (x_2, y_2)$. □

Onto. Let $(a, b) \in \mathbb{R} \times \mathbb{R}$. Take $\left(\frac{b}{2}, a - 1\right) \in \mathbb{R} \times \mathbb{R}$. Then $f\left(\left(\frac{b}{2}, a - 1\right)\right) = ((a - 1) + 1, 2 \cdot \frac{b}{2}) = (a, b)$. □

Two ordered pairs are equal exactly when both coordinates are, so one equation of pairs is **two** equations of numbers, and both must be carried through. The coordinates also cross over — the output's first entry comes from the input's second — which is why the candidate in the onto proof has b first.

Disproofs

$f : \mathbb{Z} - \{-1, 0, 1\} \to \mathbb{Z}^{>1}$, $f(x) =$ the largest prime divisor of x.

Not one-to-one: $f(10) = 5 = f(15)$ and $10 \neq 15$. Not onto: $4 \in \mathbb{Z}^{>1}$, but 4 is not prime, so no x has $f(x) = 4$ — the range of f is the set of primes, a proper subset of $\mathbb{Z}^{>1}$. Both are one line, because both are witnesses.

Is it even a function?

Before either property, the candidate has to be a function. These are examples of how that fails (functions-on-general-sets[8]):

- $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(n) = \frac{(n+1)(n+3)}{3}$ — lands **outside the co-domain** ($n = 1$ gives $\frac{8}{3}$).
- $f : \mathbb{Q} \to \mathbb{Z}$, $f(\frac{m}{n}) = m$ — **one input, two values**: $f(\frac{1}{2}) = 1$ but $f(\frac{2}{4}) = 2$, and $\frac{1}{2} = \frac{2}{4}$. A rule that depends on how the input is *written* is not well defined.
- $\{(x, y) : y^2 = x\}$ and $\{(x, y) : x = |y|\}$ — **one input, two values** again: $(4, 2)$ and $(4, -2)$; $(1, 1)$ and $(1, -1)$.

A constant function ($f(x) = 1$) and $f(n) = n!$ on $\mathbb{Z}^+$ are functions; the constant one is neither one-to-one nor onto $\mathbb{R}$, which makes it the fourth panel of Figure 1.

[8] courses/math1061/functions-on-general-sets.pdf

See also

- inverse-functions[9] — what a bijection buys you
- composition-of-functions[10] — how the two properties behave under $g \circ f$
- functions-on-general-sets[11] — the definition, co-domain versus range
- counting-selections[12] — a one-to-one function from an r-element set to an n-element set picks r different images in order, so there are $\frac{n!}{(n-r)!}$ of them
- pigeonhole-principle[13] — no function from a finite set to a smaller one is one-to-one, the pigeonhole principle
- quantified-statements[14] · proof-techniques[15] — why the templates have the shape they have
- 2026-09-07-functions-one-to-one-onto-inverse[16] — Lecture 19
- practice-problems[17] — §25

[9] courses/math1061/inverse-functions.pdf
[10] courses/math1061/composition-of-functions.pdf
[11] courses/math1061/functions-on-general-sets.pdf
[12] courses/math1061/counting-selections.pdf
[13] courses/math1061/pigeonhole-principle.pdf
[14] courses/math1061/quantified-statements.pdf
[15] courses/math1061/proof-techniques.pdf
[16] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf
[17] courses/math1061/practice-problems.pdf