# Mathematical Induction

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The principle of mathematical induction, the four-step shape every induction proof takes, and the extra care inequalities need. See sequences-and-summations[1] for the summation manipulations these proofs run on, and proof-techniques[2] for the other proof methods in the course.

### Why it works

Let $P(n)$ be a predicate written in terms of some integer $n$. Suppose you know two statements are true:

1. $P(2)$
2. $\forall k \in \mathbb{Z}$ with $k \geq 2$, if $P(k)$ then $P(k+1)$

Can you conclude $P(5)$? Yes — statement 2 is a general implication, so instantiate it repeatedly and apply modus ponens. The conclusion of one step is the premise of the next, which is what turns two facts into a chain (Figure 1).

Nothing here is special to 5. $P(5)$ is reached in three hops, but the chain never stops, so $P(n)$ is true for **each** integer $n \geq 2$.

[1] courses/math1061/sequences-and-summations.pdf
[2] courses/math1061/proof-techniques.pdf

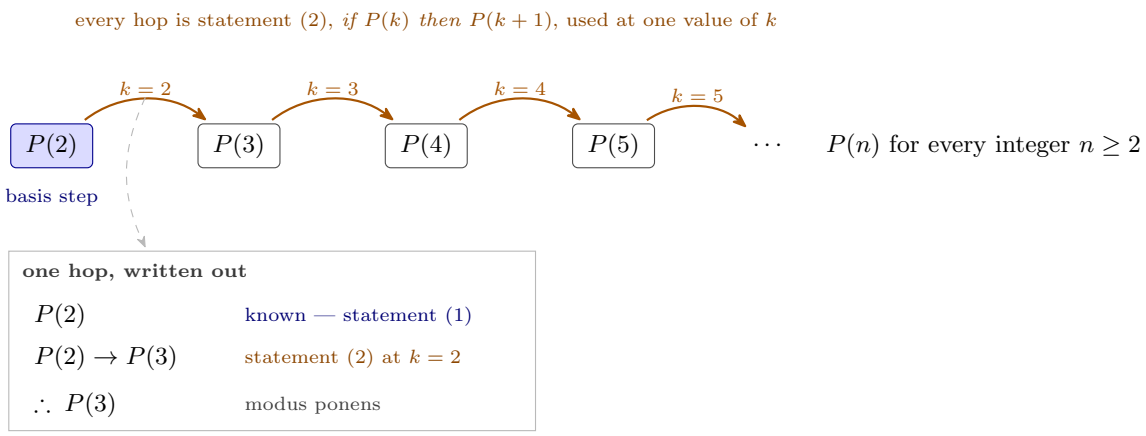



Figure 1: Why those two statements are enough. The basis step pins one statement true; every hop after it is the *same* general implication used at one more value of $k$, so each conclusion becomes the next premise. The panel writes one hop out in full — it is a single modus ponens, nothing more.

**Both statements are needed.** Statement 2 alone gives nothing: an implication $P(k) \to P(k+1)$ is true in three of the four rows of its truth table — including both rows where $P(k)$ is false. Without the basis step pinning $P$ true somewhere, the chain has nothing to start from and every link stays vacuously true while every $P(n)$ stays false.

## The shape of an induction proof

Four steps, and it is worth naming each one explicitly in the write-up so the reader can follow the argument:

1. **Define the predicate.** Let $P(n)$ denote *(the statement, with $n$ free).*
2. **Basis step.** Prove the predicate is true at the smallest value of $n$ in the claim — evaluate the two sides separately and observe they agree.
3. **Inductive hypothesis.** Suppose $k$ is an integer at or above the basis value, and that the predicate is true for $n = k$. Write out what $P(k)$ actually says.
4. **Inductive step.** Prove $P(k+1)$ follows, i.e. $\forall k \in \mathbb{Z}$ with $k \geq$ *(basis)*, if $P(k)$ is true then $P(k+1)$ is true. Start from the LHS of $P(k+1)$, use the inductive hypothesis somewhere (say where), and arrive at the RHS of $P(k+1)$.

Then conclude: *therefore, by the principle of mathematical induction, …* for each integer $n \geq$ *(basis).*

One thing about the order. In the write-up, the inductive step's goal — *prove that* $\forall k \in \mathbb{Z}$ *with* $k \geq$ *(basis), if* $P(k)$ *is true then* $P(k+1)$ *is true* — goes on the line **above** the inductive

hypothesis, and the assumption is made underneath it: goal first, assumption second. The numbering above counts the four actions; the goal line itself is not numbered. Write the goal down, then assume, then work.

### Worked example — a summation

**Prove that for each integer $n \geq 1$, $\sum_{i=1}^{n}(3i+2) = \frac{3n^2+7n}{2}$.**

*Proof.* Let $P(n)$ denote the predicate $\sum_{i=1}^{n}(3i+2) = \frac{3n^2+7n}{2}$.

**Basis step.** Prove the predicate is true for $n = 1$.

LHS of $P(1)$: $\sum_{i=1}^{1}(3i+2) = 3 \cdot 1 + 2 = 5$. RHS of $P(1)$: $\frac{3(1^2)+7(1)}{2} = \frac{3+7}{2} = 5$. Therefore $P(1)$ is true.

**Inductive step.** Prove that $\forall k \in \mathbb{Z}$ with $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

*Inductive hypothesis*: suppose $k \geq 1$ is an integer and the predicate is true for $n = k$ — that is, $\sum_{i=1}^{k}(3i+2) = \frac{3k^2+7k}{2}$.

LHS of $P(k+1)$:

$$\sum_{i=1}^{k+1}(3i+2) = \sum_{i=1}^{k}(3i+2) + (3(k+1)+2)$$

$$= \frac{3k^2+7k}{2} + 3k + 5 \qquad \text{(by the inductive hypothesis } P(k)\text{)}$$

$$= \frac{3k^2+13k+10}{2} = \frac{3(k^2+2k+1)+7k+7}{2} = \frac{3(k+1)^2+7(k+1)}{2}$$

which is the RHS of $P(k+1)$.

Therefore, by the principle of mathematical induction, $\sum_{i=1}^{n}(3i+2) = \frac{3n^2+7n}{2}$ for each integer $n \geq 1$.

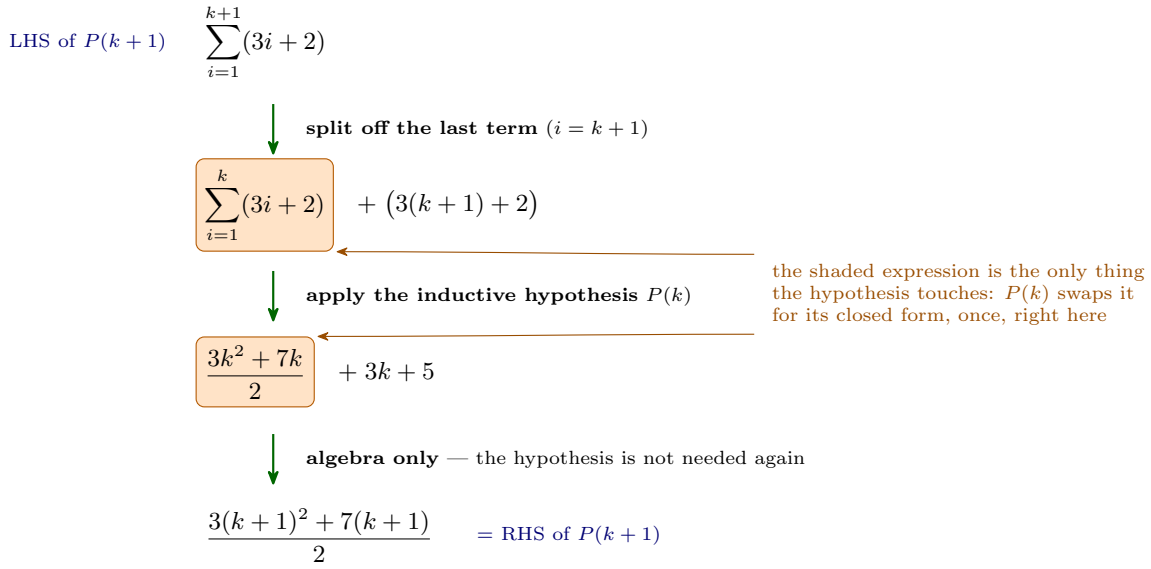



Figure 2: Where the inductive hypothesis actually enters. Splitting off the last term turns the LHS of $P(k+1)$ into *the LHS of $P(k)$, plus one more term*; the hypothesis is then spent exactly once, swapping that sum for its closed form; everything below is algebra. The shading follows that one expression from line to line.

The move at the first line is splitting off the last term (see sequences-and-summations[3]) — that is what exposes the $P(k)$ sum so the hypothesis can be applied to it. Figure 2 tracks that one expression through the chain.

Before starting the chain, write down what the RHS of $P(k+1)$ actually is — here $\frac{3(k+1)^2+7(k+1)}{2}$. Without it you are simplifying with no idea what counts as finished; with it, the last algebraic step is obvious, because you already know the shape you are steering the numerator into.

## Induction with inequalities

**Be careful when manipulating inequalities.** The rules that hold:

| Given | Conclusion |
|---|---|
| $a < b$ | $a + x < b + x$, regardless of the sign of $x$ |
| $a < b$, $x$ positive | $ax < bx$ |
| $a < b$, $x$ negative | $ax > bx$ — *the direction flips* |
| $a < b$, $x$ zero | $ax = bx$ |
| $a < b$ and $x < y$ | $a + x < b + y$ |
| $a < b$ and $x > y$ | **nothing** can be concluded about $a + x$ |

All of these still hold with $<$ and $>$ replaced by $\leq$ and $\geq$.

### Worked example — $n^2 < 2^n$

**Prove that for each integer $n \geq 5$, $n^2 < 2^n$.**

> *Proof.* Let $P(n)$ be the predicate $n^2 < 2^n$.
>
> **Basis step.** Consider $n = 5$. Then $n^2 = 25$ and $2^n = 32$. Since $25 < 32$, $P(5)$ is true.
>
> **Inductive hypothesis.** Suppose $k \geq 5$ is an integer and $P(k)$ is true. Thus $k^2 < 2^k$, and hence $2k^2 < 2 \cdot 2^k$.
>
> **Inductive step.** Consider $n = k + 1$; the goal is $(k + 1)^2 < 2^{k+1} = 2 \cdot 2^k$.
>
> $$(k+1)^2 = k^2 + 2k + 1 < k^2 + 2k + k = k^2 + 3k < k^2 + k^2 = 2k^2 < 2(2^k) = 2^{k+1}$$
>
> The first inequality uses $1 < k$ (since $k \geq 5$); the second uses $3k < k \cdot k$, which holds because $5 \leq k$; the last uses the inductive hypothesis. Thus $P(k + 1)$ is true.

[3] courses/math1061/sequences-and-summations.pdf

Therefore, by PMI, for each integer $n \geq 5$, $n^2 < 2^n$.

The pattern for an inequality is a **chain**: start at the LHS of $P(k+1)$ and relax it one step at a time — each step justified by a rule from the table or by the inductive hypothesis — until it reaches the RHS of $P(k+1)$. Every step must be an inequality in the *same* direction, or the chain says nothing.

### The inductive step must hold at every $k$

Including the first one. An inductive step that is sound for all large $k$ but fails at the basis value leaves a gap the chain cannot cross, and the induction proves nothing beyond the basis case — even though every individual line of the argument can look correct.

The classic demonstration is the 'proof' that all numbers are equal, whose inductive step is valid for every $k \geq 2$ and false at $k = 1$; it is picked apart in 2026-08-31-recursive-definitions[4]. When checking an induction proof, run the inductive step at the smallest $k$ it claims to cover and see whether it still says anything.

### Strong induction

When the inductive step needs more than just the immediately preceding case — typically because a recurrence reaches back several terms, or because the step splits a number into two smaller ones it cannot control — ordinary induction is not enough. See strong-induction-and-the-well-ordering-principle[5] for the stronger form, how many base cases it demands, and the well-ordering principle it is equivalent to.

### See also

- sequences-and-summations[6] · strong-induction-and-the-well-ordering-principle[7] · proof-techniques[8]
- solving-recurrence-relations[9] — induction used to prove an explicit formula for a recursively defined sequence
- inclusion-exclusion[10] — the rule for $n$ sets, from the rule for two by splitting off the last set, an induction on $n$

[4] courses/math1061/2026-08-31-recursive-definitions.pdf
[5] courses/math1061/strong-induction-and-the-well-ordering-principle.pdf
[6] courses/math1061/sequences-and-summations.pdf
[7] courses/math1061/strong-induction-and-the-well-ordering-principle.pdf
[8] courses/math1061/proof-techniques.pdf
[9] courses/math1061/solving-recurrence-relations.pdf
[10] courses/math1061/inclusion-exclusion.pdf

- 2026-08-24-sequences-and-mathematical-induction[11] — Lecture 13
- practice-problems[12] — §17 Mathematical Induction, with full worked solutions

[11] courses/math1061/2026-08-24-sequences-and-mathematical-induction.pdf
[12] courses/math1061/practice-problems.pdf