Inverse Functions

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When a function can be run backwards, what the backwards function is, and the one notational trap: f^{-1} **means two different things**, and the braces tell you which. Builds on one-to-one-and-onto-functions[1]; the preimage half is from functions-on-general-sets[2]; the lecture is 2026-09-07-functions-one-to-one-onto-inverse[3].

Bijections have inverses

A function that is both one-to-one and onto is said to be a **bijection**.

A bijection $f : X \to Y$ has an **inverse function** $f^{-1} : Y \to X$, where for all $y \in Y$,

$$f^{-1}(y) = x \text{ where } x \text{ is the unique } x \in X \text{ such that } f(x) = y.$$

Note the direction: $f : X \to Y$ **gives** $f^{-1} : Y \to X$. The inverse's domain is f's co-domain and vice versa.

The definition is exactly one sentence, and both properties are inside it:

- "x … *such that* $f(x) = y$" — such an x **exists** for every y because f is **onto**.
- "*the unique* x" — there is only **one** because f is **one-to-one**.

[1] courses/math1061/one-to-one-and-onto-functions.pdf
[2] courses/math1061/functions-on-general-sets.pdf
[3] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf

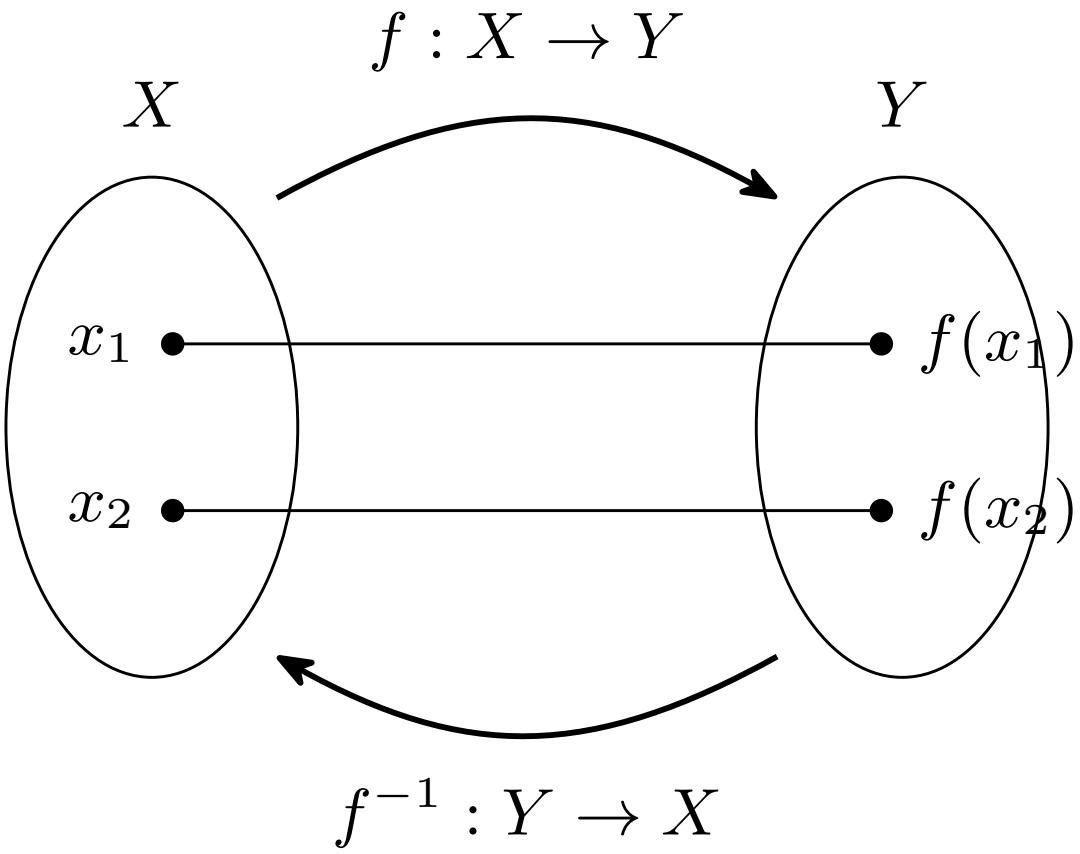


Figure 1: A bijection f pairs x_1 with $f(x_1)$ and x_2 with $f(x_2)$, one arrow each, nothing shared, nothing missed — and so the arrows can be read backwards. The curved arrow above is $f : X \to Y$; the one below is $f^{-1} : Y \to X$, the same pairs in the other direction.

Drop either and f^{-1} is not a function: if f is not onto, some y has nowhere to go; if f is not one-to-one, some y has two places to go. Both failures are drawn in 2026-09-07-functions-one-to-one-onto-inverse[4]. And the converse holds too:

$$f \text{ has an inverse} \iff f \text{ is a bijection.}$$

Finding the inverse

The onto proof computes it. Proving f onto means taking an arbitrary y (or (a, b)) and producing the x that maps to it, as a formula in y. That formula is f^{-1}. It is a function whether or not f is one-to-one; one-to-one is what makes it the *inverse*, by guaranteeing the x it produces is the only preimage, so that $f^{-1} \circ f$ is the identity and not merely $f \circ f^{-1}$.

> $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $f((x, y)) = (y + 1, 2x)$ — a bijection (proved in one-to-one-and-onto-functions[5]).

The onto proof took, for (a, b), the candidate $\left(\frac{b}{2}, a - 1\right)$. So

$$f^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \qquad f^{-1}((a, b)) = \left(\tfrac{b}{2},\ a - 1\right), \quad \text{i.e.} \quad f^{-1}((x, y)) = \left(\tfrac{y}{2},\ x - 1\right).$$

Sense check, both ways:

$$f(f^{-1}((x, y))) = f(\left(\tfrac{y}{2}, x - 1\right)) = \left((x - 1) + 1,\ 2 \cdot \tfrac{y}{2}\right) = (x, y),$$

$$f^{-1}(f((x, y))) = f^{-1}((y + 1, 2x)) = \left(\tfrac{2x}{2},\ (y + 1) - 1\right) = (x, y).$$

In the language of composition-of-functions[6]: $f \circ f^{-1}$ and $f^{-1} \circ f$ are both the identity, and checking one of them is the standard way to confirm an inverse has been computed correctly.

For a function given by a formula in one variable the procedure is the same: set $f(x) = y$, solve for x in terms of y, and the solution is $f^{-1}(y)$ — provided the solution is unique and always in the domain, which is what "bijection" guarantees.

[4] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf
[5] courses/math1061/one-to-one-and-onto-functions.pdf
[6] courses/math1061/composition-of-functions.pdf

The two meanings of f^{-1}

This is the most common source of wrong answers in this topic.

1. Preimage. If $f : X \to Y$ is **any** function and $A \subseteq Y$, then the **inverse image** or **preimage** of A is

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

The preimage function is a function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$.

2. Inverse function. If $f : X \to Y$ is a **bijection**, then the **inverse function** of f is $f^{-1} : Y \to X$ with $f^{-1}(y) = $ the unique x such that $f(x) = y$. The inverse function is a function $f^{-1} : Y \to X$.

	Takes	Gives	Exists
preimage $f^{-1}(A)$	a **subset** $A \subseteq Y$	a **subset** of X	for **every** function
inverse function $f^{-1}(y)$	an **element** $y \in Y$	an **element** of X	only if f is a **bijection**

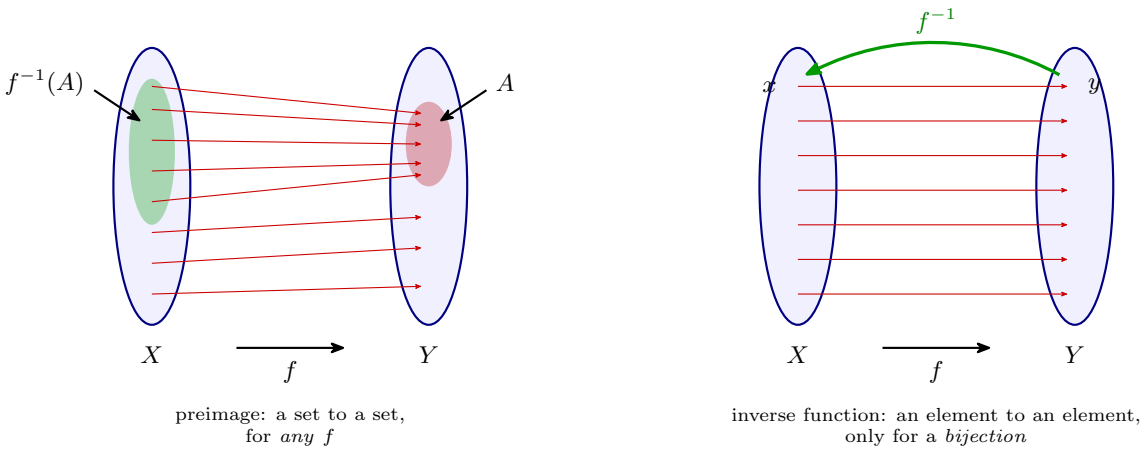


Figure 2: The two meanings side by side. Left: a many-to-one f, with a subset A of Y (red) and its preimage $f^{-1}(A)$ (green) — every x whose arrow lands in A. This exists for any function, and it is a set. Right: a bijective f, every arrow its own, and the single green arrow f^{-1} running one element back from y to x. Same symbol; the first eats sets and the second eats elements.

The braces decide. $f^{-1}(\{a\})$ has a set inside, so it is a preimage and always means something. $f^{-1}(a)$ has an element inside, so it is the inverse function and means something only if f is a bijection.

Worked: reading the braces

$f : \{1, 2, 3, 4\} \to \{a, b, c\}$, $f = \{(1, a), (2, a), (3, b), (4, b)\}$.

f is not one-to-one ($1, 2 \mapsto a$) and not onto (c unreached), so it is not a bijection.

Statement		Because
The inverse function f^{-1} does not exist.	**T**	f is not a bijection
$f^{-1}(\{a\}) = \{1, 2\}$.	**T**	preimage: 1 and 2 map to a, nothing else does
$f^{-1}(a)$ is not defined.	**T**	a is an element, so this is the inverse function, which does not exist
$f^{-1}(\{c\}) = \emptyset$.	**T**	preimage: nothing maps to c, and $\emptyset$ is a perfectly good answer

All four are true. The second and third differ *only* in the braces, and one is a set while the other is undefined.

When f **is** a bijection the two meanings agree in the obvious way: $f^{-1}(\{y\}) = \{f^{-1}(y)\}$, a one-element set whose element is the inverse function's value. That is the only case in which it is safe to blur them, and even then the types differ.

See also

- one-to-one-and-onto-functions[7] — the two properties a bijection needs, and how to prove them
- composition-of-functions[8] — $f \circ f^{-1}$ and $f^{-1} \circ f$
- symmetric-group[9] — the inverse function as the inverse of an element of a group
- groups[10] — $f(x) = ax + b$ with $a \neq 0$ has inverse $f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}$, its inverse in the group of such functions
- functions-on-general-sets[11] — image and preimage, where meaning 1 comes from
- sets-and-set-notation[12] — power sets, which the preimage function maps between

[7] courses/math1061/one-to-one-and-onto-functions.pdf
[8] courses/math1061/composition-of-functions.pdf
[9] courses/math1061/symmetric-group.pdf
[10] courses/math1061/groups.pdf
[11] courses/math1061/functions-on-general-sets.pdf
[12] courses/math1061/sets-and-set-notation.pdf

- 2026-09-07-functions-one-to-one-onto-inverse[13] — Lecture 19 · 2026-09-08-composition-of-functions[14] — Lecture 20
- practice-problems[15] — §25

[13] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf
[14] courses/math1061/2026-09-08-composition-of-functions.pdf
[15] courses/math1061/practice-problems.pdf