Incidence Matrices

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An incidence matrix writes a graph down by vertices against edges: a row for each vertex, a column for each edge, and in each place the number of times that vertex is incident with that edge. Each column is one edge, so the matrix is in effect the table of endpoints written as numbers. It has a simple shape that makes it easy to check: every column adds to 2, because every edge has two ends, and every row adds to the degree of its vertex. Putting those together proves the Handshake Theorem again. The other matrix of a graph, vertices against vertices, is in adjacency-matrices[1]. Graphs, loops, degree and the Handshake Theorem are in graphs[2]. First met in 2026-10-27-matrix-representations-of-graphs[3].

Definition

> The **incidence matrix** of a graph with vertices $v_1, v_2, \dots, v_n$ and edges $e_1, e_2, \dots, e_m$ is the $n \times m$ matrix $N = [n_{ij}]$ where each entry n_{ij} is the number of times vertex v_i is incident with edge e_j.

Rows are vertices and columns are edges. The entry n_{ij} is about vertex v_i and edge e_j, in that order, and the first index is the row. The matrix is $n \times m$: as many rows as vertices, as many columns as edges. Swapping the roles gives the transpose N^{T}, which holds the same

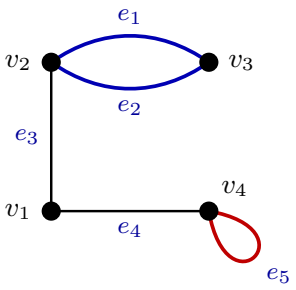
[1] courses/math1061/adjacency-matrices.pdf
[2] courses/math1061/graphs.pdf
[3] courses/math1061/2026-10-27-matrix-representations-of-graphs.pdf

information but is not the incidence matrix as this course defines it, so write the vertices down the side and the edges across the top, and label both. Like the adjacency matrix, N depends on the order chosen for the vertices and for the edges.

An edge has two ends. For a vertex v_i and an edge e_j there are three cases:

- e_j does not have v_i as an endpoint: $n_{ij} = 0$.
- e_j joins v_i to a different vertex: one end is at v_i, so $n_{ij} = 1$.
- e_j is a loop at v_i: both ends are at v_i, so $n_{ij} = 2$.

So **every entry is 0, 1 or 2**, and **a 2 means a loop**.



$N =$

	e_1	e_2	e_3	e_4	e_5	sum
v_1	0	0	1	1	0	2
v_2	1	1	1	0	0	3
v_3	1	1	0	0	0	2
v_4	0	0	0	1	**2**	3
sum	2	2	2	2	2	

Figure 1: A graph and its incidence matrix, one row for each vertex and one column for each edge. Each column is an edge: e3 has a 1 at each of its endpoints v1 and v2, and the loop e5, in red, has a single 2 at v4, because it meets v4 at both of its ends. The parallel edges e1 and e2, in blue, have the same endpoints and so identical columns. Every column adds to 2, and each row adds to the degree of its vertex: 2, 3, 2, 3.

Columns are edges

The column of e_j lists how many times each vertex meets e_j. So:

- **Every column adds to 2.** An edge has exactly two ends, and each end is at one vertex. A column is either **two 1s** (an edge with two different endpoints, the rows of the 1s) or **a single 2** (a loop, at the row of the 2), and 0 everywhere else.
- **Parallel edges have identical columns.** Two edges with the same endpoints meet every vertex the same number of times. In the figure, e_1 and e_2 are both $(0, 1, 1, 0)$.
- **A column is never all zeros**, since every edge has at least one endpoint.

Rows are vertices

The row of v_i lists how many times v_i meets each edge.

- **The row sum is the degree.** Adding the row counts the ends of edges at v_i, a loop's two ends included, which is $\deg(v_i)$ (graphs[4]). Unlike the adjacency matrix, no correction for loops is needed: the loop's 2 is already in the row.
- **An isolated vertex has a row of zeros.** It still gets its row.

The Handshake Theorem again

Add up every entry of N in two ways.

- **Column by column:** each of the m columns adds to 2, so the total is $2m$.
- **Row by row:** row i adds to $\deg(v_i)$, so the total is $\deg(v_1) + \cdots + \deg(v_n)$.

The two totals are the same number, so

$$\sum_{i=1}^{n} \deg(v_i) = 2m,$$

which is the Handshake Theorem (graphs[5]). It is the same proof as counting edge ends, with the ends laid out in a table: each entry counts ends of one edge at one vertex. In the figure, the row sums $2+3+2+3$ and the column sums $2+2+2+2+2$ both come to 10.

Loops

A loop at v_i is incident with v_i **twice**, once at each of its ends, so its column has a **2** in row v_i and 0 everywhere else. In the figure the loop e_5 is the column $(0, 0, 0, 2)$.

This is what keeps the column sums at 2 and makes the row sums the degrees: the 2 is the loop's contribution to $\deg(v_i)$.

The adjacency matrix counts a loop differently. It counts edges, and a loop is one edge, so it puts a **1** on the diagonal (adjacency-matrices[6]). The same loop is a 1 in A and a 2 in N. A column with a single 1 and nothing else is not a loop. It is not an edge at all, since it has only one end.

[4] courses/math1061/graphs.pdf
[5] courses/math1061/graphs.pdf
[6] courses/math1061/adjacency-matrices.pdf

Reading a graph off a matrix

Read an incidence matrix **one column at a time**:

1. Draw a dot for each row and label it $v_1, \ldots, v_n$.
2. For each column e_j: if it has 1s in rows i and k, draw an edge e_j joining v_i and v_k; if it has a 2 in row i, draw a loop e_j at v_i.
3. Check: every column added to 2, and each vertex's degree in the drawing is its row sum.

For example,

$$N = \begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline v_1 & 1 & 0 & 0 & 2 & 0 \\ v_2 & 1 & 1 & 1 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

has every column adding to 2. Column by column: e_1 joins v_1 and v_2; e_2 and e_3 both join v_2 and v_3, identical columns, so they are parallel; e_4 is a loop at v_1; e_5 joins v_3 and v_4.

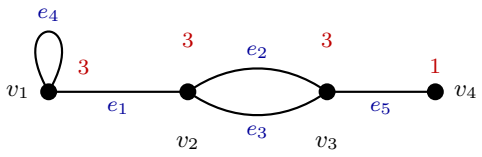


Figure 2: The graph read off the incidence matrix, one column at a time. Columns e2 and e3 are identical, so those two edges are parallel, and column e4 is a single 2 in row v1, so it is a loop at v1. The degrees, in red, are the row sums of the matrix, 3, 3, 3, 1, and they add to 10, twice the 5 edges.

Unlike an adjacency matrix, an incidence matrix names every edge, so the graph comes back with its edge names as well.

Which matrices are incidence matrices

A matrix with at least one row is the incidence matrix of some graph if and only if every entry is a non-negative integer and **every column adds to 2**. Equivalently, every column is either two 1s and the rest 0, or a single 2 and the rest 0.

The two forms agree: non-negative integers adding to 2 are either $1+1$ or 2, with zeros elsewhere.

Proof. ($\Rightarrow$) Shown above: each column is an edge, and an edge is two 1s at its two endpoints or a single 2 at the vertex of a loop.

($\Leftarrow$) Let M be an $n \times m$ matrix, $n \geq 1$, whose columns each have one of the two forms. Build a graph with vertices $v_1, \ldots, v_n$ and, for each column j, an edge e_j: if column j has 1s in rows i and k, let e_j join v_i and v_k; if it has a 2 in row i, let e_j be a loop at v_i. Then v_i is incident with e_j once if v_i is one of two endpoints, twice if e_j is a loop at v_i, and not at all otherwise, which is the entry m_{ij} in every case. So M is this graph's incidence matrix. $\square$

Only the columns are tested. The rows can be anything, a row of zeros included (an isolated vertex), and the matrix does not have to be square or symmetric. A graph with no edges has no columns at all, and then there is nothing to check.

Matrix	Column sums	Incidence matrix?
$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$	2, 2	yes: an edge v_1v_2 and a loop at v_1
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$	2, 1	no: the second edge has only one end
$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	3	no: an edge with three ends
$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$	2	yes: three vertices, a loop at v_2, and v_1, v_3 isolated

Four more are worked in 2026-10-27-matrix-representations-of-graphs[7].

Adjacency and incidence compared

The graph in the figure has both matrices:

$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 2 & 0 \\ v_3 & 0 & 2 & 0 & 0 \\ v_4 & 1 & 0 & 0 & 1 \end{array} \qquad N = \begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline v_1 & 0 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 1 & 1 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 & 2 \end{array}$$

[7] courses/math1061/2026-10-27-matrix-representations-of-graphs.pdf

	Adjacency matrix A	Incidence matrix N
Size	$n \times n$	$n \times m$
Rows	vertices	vertices
Columns	vertices	edges
Entry	number of edges with endpoints $\{v_i, v_j\}$	number of times v_i is incident with e_j
Entries can be	any non-negative integer	0, 1 or 2
A loop at v_i	1 on the diagonal, at a_{ii}	2 in its column, at row v_i
Parallel edges	one entry bigger than 1	identical columns
Shape	square and symmetric	any; each column adds to 2
$\deg(v_i)$	row sum plus a_{ii}	row sum
Number of edges	sum on and above the diagonal	number of columns
Edge names	not recorded	one column per named edge

From N to A. Off the diagonal, a_{ij} is the number of columns with a 1 in both row i and row j. On the diagonal, a_{ii} is the number of columns with a 2 in row i. In N above, two columns (e_1 and e_2) have 1s in rows v_2 and v_3, so $a_{23} = 2$, and one column (e_5) has a 2 in row v_4, so $a_{44} = 1$.

From A to N, the edges first have to be named, since A only says how many edges join each pair. Any naming gives an incidence matrix of the same graph, with its columns in a different order.

A is the more compact when there are many edges. N is the one to use when the edges themselves matter, as in listing an Euler circuit edge by edge, which is done from an incidence matrix in 2026-10-27-matrix-representations-of-graphs[8].

Common mistakes

- **Swapping rows and columns.** In this course the vertices are the rows and the edges are the columns. The transpose has the same information but is not the incidence matrix as defined here.
- **Writing 1 for a loop.** A loop meets its vertex twice, so it is a 2. A column with a single 1 has only one end and is not an edge.
- **Adding a loop correction to the row sum.** That is for the adjacency matrix. In N the row sum already is the degree.
- **Checking the rows, or symmetry, instead of the columns.** Whether a matrix is an incidence matrix is decided by its columns alone: each must add to 2.
- **Leaving out an isolated vertex.** Every vertex has a row, even if it is all zeros.

[8] courses/math1061/2026-10-27-matrix-representations-of-graphs.pdf

- **Leaving off the labels.** The matrix depends on the order of the vertices and of the edges.

See also

- adjacency-matrices[9] — the vertex-by-vertex matrix, where a loop counts 1
- graphs[10] — loops, parallel edges, degree and the Handshake Theorem
- euler-circuits-and-trails[11] — the even-degree test for an Euler circuit
- 2026-10-27-matrix-representations-of-graphs[12] — Lecture 35
- practice-problems[13] — §45, with full worked solutions

[9] courses/math1061/adjacency-matrices.pdf
[10] courses/math1061/graphs.pdf
[11] courses/math1061/euler-circuits-and-trails.pdf
[12] courses/math1061/2026-10-27-matrix-representations-of-graphs.pdf
[13] courses/math1061/practice-problems.pdf