Functions on General Sets

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The set-theoretic definition of a function — not "a rule" or "a formula", but a particular kind of **subset of a Cartesian product**. See sets-and-set-notation[1] for the Cartesian products this is built on.

The definition

> A **function** $f : X \to Y$ is a subset $f \subseteq X \times Y$ such that for each $x \in X$, there is a **unique** $y \in Y$ such that $(x, y) \in f$.

The function *is* the set of pairs. There is no separate "rule" object — the familiar notations are all names for the same set:

$$y = 2x + 1 \qquad \text{or} \qquad f(x) = 2x + 1 \qquad \text{or} \qquad f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 2x + 1\}$$

and the bridge between the two styles is

$$f(x) = y \quad \iff \quad (x, y) \in f$$

Defining a function this way costs a little intuition and buys a great deal of precision: **the definition applies to any sets at all**, not just sets of numbers, and questions like "are these two functions equal?" become ordinary questions about set equality.

[1] courses/math1061/sets-and-set-notation.pdf

Three conditions hidden in one sentence

Unpacking "for each $x \in X$ there is a unique $y \in Y$ with $(x, y) \in f$" gives three separate things to check:

1. **Every** $x \in X$ is used — no element of the domain is left without a value. ("for each $x \in X$")
2. Every value lands **in** Y — no pair points somewhere outside the codomain.
3. Each x has **only one** value — no x appears as the first coordinate of two different pairs. ("unique")

Checking a candidate means checking all three, and a candidate can fail each one separately.

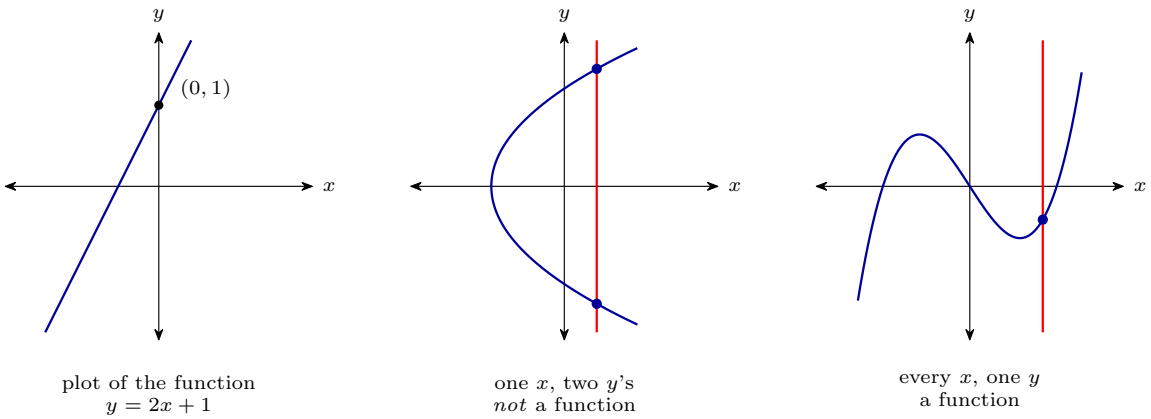


Figure 1: The uniqueness condition, seen graphically. A subset of the plane is a function of x exactly when every vertical line meets it at most once: the middle curve fails because one input is paired with two outputs, while the right-hand curve passes despite repeating output values, which the definition never forbids. The red vertical line in the middle and right-hand panels is the test.

The asymmetry in that picture is the whole definition: **repeated y-values are fine, repeated x-values are not.** The right-hand curve takes the same value three times over and is still a function.

Worked: which of these is a function?

Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3, 4\}$. Which one of the following is a function from X to Y?

	Candidate	Why
(a)	$\{(1,2),(1,3),(2,4)\}$	$f(1)=2$ *and* $f(1)=3$ — the value at 1 is **not unique**. (Also 3 is unused.)
(b)	$\{(1,2),(2,3),(3,2),(4,1)\}$	$4 \notin X$ — a pair whose first coordinate is outside the domain
(c)	$\{(1,2),(2,3),(3,5)\}$	$5 \notin Y$ — a value outside the codomain
(d)	$\{(1,2),(2,1),(3,1)\}$	all first coordinates in X, all second in Y, each of $1,2,3$ used exactly once

Each wrong answer breaks a **different** one of the three conditions, which is what makes this a good question: the three failures are not variations on one mistake.

For (d): $f(1)=2$, $f(2)=1$, $f(3)=1$. Note $f(2)=f(3)$ — two inputs sharing a value is not a problem, for exactly the reason the picture above shows.

Codomain and range

The two are constantly confused, and the difference is one word:

> For a function $f : X \to Y$, the **co-domain** is Y, and the **range** is $f(X) = \{f(x) : x \in X\}$, which is a **subset of** Y.

	Is	Determined by
Co-domain	Y — where values are *allowed* to land	the declaration $f : X \to Y$
Range	$f(X)$ — where values *actually* land	the pairs in f

So the codomain is a choice you make when you write down the function, and the range is a fact you compute. Always $f(X) \subseteq Y$, and the containment is often proper.

For the function (d) above, $f = \{(1,2),(2,1),(3,1)\}$:

$$\text{co-domain} = Y = \{1,2,3,4\} \qquad \text{range} = f(X) = \{1,2\} \subseteq Y$$

3 and 4 are in the codomain and not in the range — they are permitted values that this particular function never takes.

Image and preimage

The two definitions generalise "range" to any subset, in each of the two directions:

> If $f : X \to Y$ is **any** function and A is any subset of X, then the **image** of A is
>
> $$f(A) = \{f(x) : x \in A\}$$
>
> If $f : X \to Y$ is **any** function and B is any subset of Y, then the **inverse image** or **preimage** of B is
>
> $$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Both take a **set** and return a **set** — $A \subseteq X$ gives $f(A) \subseteq Y$, and $B \subseteq Y$ gives $f^{-1}(B) \subseteq X$. The range is then just the image of the whole domain, $f(X)$.

> **The notation f^{-1} here does not mean an inverse function.** $f^{-1}(B)$ is defined for **any** function, invertible or not — the emphasis on *any* in both definitions is the point. It is a set-to-set operation that always exists; an inverse function is a different thing that usually does not.

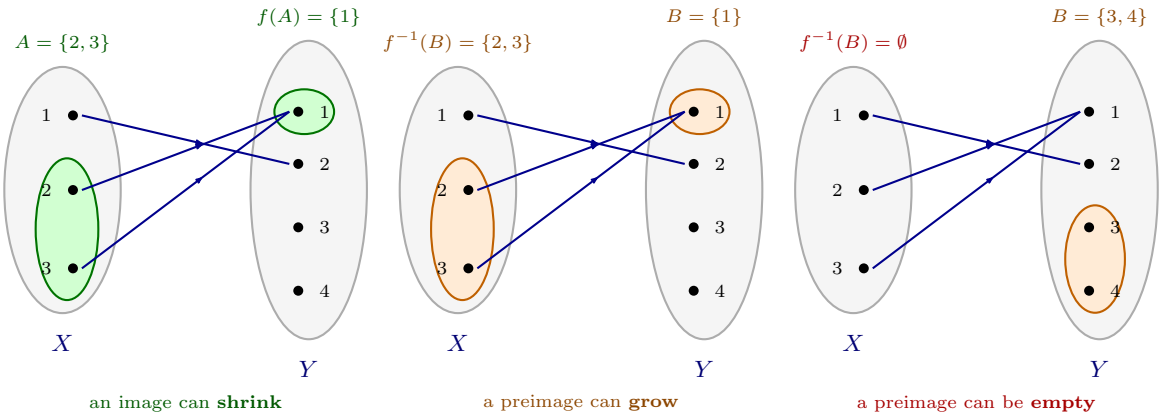


Figure 2: Image and preimage of the same function, read in opposite directions. Following the lines forward from a subset of X collects a subset of Y; following them backwards from a subset of Y collects every x that lands in it — which can be several elements, or none at all. In each panel the domain is on the left and the co-domain on the right, each line runs from an element of X to its value in Y, and the subset under discussion is outlined in colour with its expression in the same colour.

Worked

Let $X = \{1, 2, 3\}$, $Y = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 1), (3, 1)\}$, the function from the question above. So $f(1) = 2$, $f(2) = 1$, $f(3) = 1$.

	Asked for	Working	Answer
(a)	image of $\{1\}$	$f(1) = 2$	$f(\{1\}) = \{2\}$
(b)	image of $\{2, 3\}$	$f(2) = 1$, $f(3) = 1$	$f(\{2, 3\}) = \{1\}$
(c)	inverse image of $\{2\}$	which x have $f(x) = 2$? only $x = 1$	$f^{-1}(\{2\}) = \{1\}$
(d)	preimage of $\{1\}$	which x have $f(x) = 1$? $x = 2$ and $x = 3$	$f^{-1}(\{1\}) = \{2, 3\}$
(e)	preimage of $\{3, 4\}$	no x maps to 3 or 4	$f^{-1}(\{3, 4\}) = \emptyset$

Three things this table is designed to show — they are the three panels of Figure 2:

- **(b): an image can be smaller than the set you started with.** $\{2, 3\}$ has two elements, $f(\{2, 3\})$ has one, because f sends both to the same place.
- **(d): a preimage can be larger than the set you started with.** One element of Y pulls back to two of X.
- **(e): a preimage can be empty.** $\{3, 4\}$ lies outside the range entirely. Note this is a perfectly good answer, not a failure — $\emptyset$ is a subset of X like any other, and it is the reason f^{-1} needs no invertibility to be defined.

Always write the answers as **sets**: $f^{-1}(\{2\}) = \{1\}$, not $= 1$. The operation returns a subset, and dropping the braces is the standard slip.

See also

- sets-and-set-notation[2] — Cartesian products, and the $\in$ / $\subseteq$ distinction these definitions lean on
- proving-set-properties[3] — the proof methods, for arguing about images and preimages
- relations-on-sets[4] — functions as the relations with exactly one arrow out of each element
- 2026-09-03-set-properties-and-functions[5] — Lecture 18
- practice-problems[6] — §24 Functions Defined on General Sets, with full worked solutions

[2] courses/math1061/sets-and-set-notation.pdf
[3] courses/math1061/proving-set-properties.pdf
[4] courses/math1061/relations-on-sets.pdf
[5] courses/math1061/2026-09-03-set-properties-and-functions.pdf
[6] courses/math1061/practice-problems.pdf