Composition of Functions

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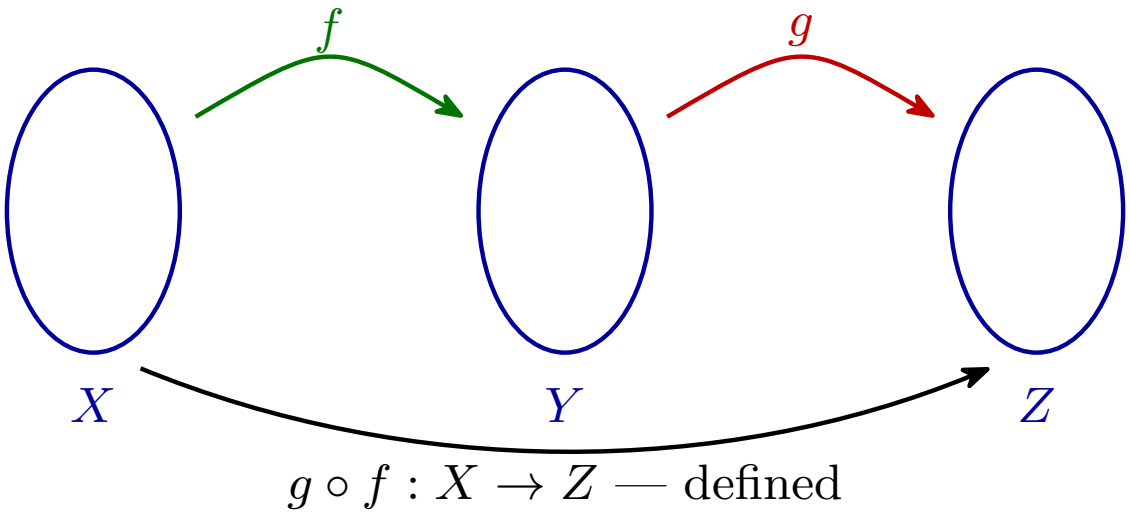
Chaining two functions into one, and which properties survive the chaining — in both directions. First met in 2026-09-08-composition-of-functions[1]; the properties are defined in one-to-one-and-onto-functions[2] and inverse-functions[3].

Definition

> Let $f : X \to Y$ and $g : Y \to Z$ be functions. The **composition** of f and g is the function $g \circ f : X \to Z$ defined by
>
> $$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X.$$

Three things to read off the definition:

- **Right to left.** $g \circ f$ applies f *first*, then g. The order of the symbols matches the formula $g(f(x))$, not the order of events.
- **The domain of $g \circ f$ is the domain of f; the co-domain is the co-domain of g.** The middle set Y vanishes from the type.
- **It is defined only when f's co-domain is g's domain.** $g \circ f$ makes sense because f lands in Y and g eats Y. The other composite, $f \circ g$, would need g's outputs (in Z) to be inputs to f (from X), so it exists only if $Z = X$.



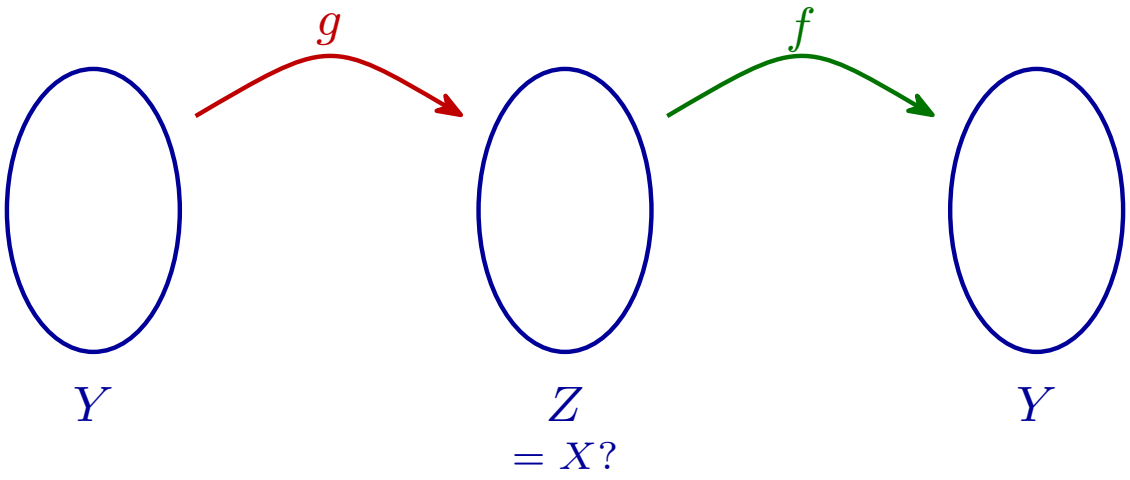


Figure 1: Why $g \circ f$ exists and $f \circ g$ usually does not. Top: f then g chains because f lands exactly where g starts. Bottom: g then f would need g's outputs, in Z, to be acceptable inputs to f, which come from X — so it is defined only if $Z = X$.

The composition itself is drawn in 2026-09-08-composition-of-functions[4]: three ellipses, $x \mapsto f(x) \mapsto g(f(x))$, with the long arrow $g \circ f$ beneath.

Composition is not commutative. Even when both orders exist ($X = Y = Z$), $g \circ f$ and $f \circ g$ are generally different functions: with $f(x) = x+1$ and $g(x) = 2x$ on $\mathbb{R}$, $(g \circ f)(x) = 2x+2$ but $(f \circ g)(x) = 2x + 1$.

Inverses and composition

For a bijection $f : X \to Y$ with inverse $f^{-1} : Y \to X$ (inverse-functions[5]), both composites exist and both do nothing:

$$(f^{-1} \circ f)(x) = x \text{ for all } x \in X, \qquad (f \circ f^{-1})(y) = y \text{ for all } y \in Y.$$

This is the **sense check** on a computed inverse: for $f((x, y)) = (y + 1, 2x)$ and the claimed $f^{-1}((x, y)) = (\frac{y}{2}, x - 1)$,

$$f(f^{-1}((x, y))) = f((\tfrac{y}{2}, x - 1)) = (x, y). \checkmark$$

Which properties compose

Two facts:

> If f and g are both **onto**, then $g \circ f$ is onto. If f and g are both **one-to-one**, then $g \circ f$ is one-to-one.

So both bijective gives $g \circ f$ bijective. The proofs are short:

- *Onto.* Let $z \in Z$. Since g is onto, $\exists\, y \in Y$ with $g(y) = z$. Since f is onto, $\exists\, x \in X$ with $f(x) = y$. Then $g(f(x)) = z$. $\square$
- *One-to-one.* Suppose $g(f(x_1)) = g(f(x_2))$. Since g is one-to-one, $f(x_1) = f(x_2)$. Since f is one-to-one, $x_1 = x_2$. $\square$

[1] courses/math1061/2026-09-08-composition-of-functions.pdf
[2] courses/math1061/one-to-one-and-onto-functions.pdf
[3] courses/math1061/inverse-functions.pdf
[4] courses/math1061/2026-09-08-composition-of-functions.pdf
[5] courses/math1061/inverse-functions.pdf

Which properties come back

The converse asks which ingredient must have a property when the **composite** has it. The answer is asymmetric:

If $g \circ f$ is …	then f must be …	then g must be …
onto	*not necessarily*	**onto**
one-to-one	**one-to-one**	*not necessarily*

The property comes back to the function on its own side. Onto is about covering Z, and g is the function that lands in Z. One-to-one is about not merging elements of X, and f is the function that leaves X. The other ingredient can misbehave in the middle set Y without the composite noticing, as long as the misbehaviour is never exercised.

The two theorems, by contraposition

If $g \circ f$ is onto, then g is onto.

Proof. We prove the contrapositive: if g is not onto then $g \circ f$ is not onto. Suppose g is not onto. Then $\exists\, z \in Z$ not in the range of g. Then no $x \in X$ has $g(f(x)) = z$ — for if one did, $y = f(x)$ would have $g(y) = z$, a contradiction. So z is not in the range of $g \circ f$, and $g \circ f$ is not onto. □

If $g \circ f$ is one-to-one, then f is one-to-one.

Proof. We prove the contrapositive: if f is not one-to-one then $g \circ f$ is not one-to-one. Suppose f is not one-to-one. Then $\exists\, x_1 \neq x_2 \in X$ with $f(x_1) = f(x_2)$. Hence $g(f(x_1)) = g(f(x_2))$, i.e. $(g \circ f)(x_1) = (g \circ f)(x_2)$ with $x_1 \neq x_2$, so $g \circ f$ is not one-to-one. □

Both go by contraposition (proof-techniques[6]) for the same reason: *not onto* and *not one-to-one* are **existential** and hand you a concrete witness — a missed z, a colliding pair — to push through the composite. The positive forms are universal and give nothing to hold onto. Anything the composite reaches, g reaches; anything f merges, the composite merges.

[6] courses/math1061/proof-techniques.pdf

The two counterexamples, to order

> **$g \circ f$ onto does not force f onto.** $X = \{a, b\}$, $Y = \{0, 1, 2\}$, $Z = \{v, w\}$; $f = \{(a, 0), (b, 1)\}$, $g = \{(0, v), (1, w), (2, w)\}$. Then f misses 2, but $g \circ f = \{(a, v), (b, w)\}$ is onto.

> **$g \circ f$ one-to-one does not force g one-to-one.** $X = \{a, b\}$, $Y = \{1, 2, 3\}$, $Z = \{u, v\}$; $f = \{(a, 1), (b, 2)\}$, $g = \{(1, u), (2, v), (3, v)\}$. Then $g(2) = g(3)$, but $g \circ f = \{(a, u), (b, v)\}$ is one-to-one.

Both are drawn in 2026-09-08-composition-of-functions[7], as (1) and (4) there, and both are built by the same recipe, which is the transferable skill:

1. **Make Y one element bigger than it needs to be.** That spare element is where the defect will live.
2. **Give the defect to the function on the wrong side.** For (1) f must *miss* the spare element; for (4) g must *merge* the spare element with another.
3. **Make sure the composite never touches it.** In (1), g sends the missed element somewhere already covered, so the gap is invisible from Z. In (4), f never lands on the merged element, so the collision is never fed.
4. **Use small sets.** Two-element X and Z with a three-element Y, as above, is enough; even smaller works. A counterexample is a witness; it does not need to be interesting.

The mirror symmetry is exact: (1) is *gap in f, hidden by a merge in g*; (4) is *merge in g, hidden by a gap in f*.

See also

- one-to-one-and-onto-functions[8] — the properties and their proof templates
- inverse-functions[9] — $f \circ f^{-1}$ as the identity
- symmetric-group[10] — composition as the operation of a group, on the bijections of $\{1, \ldots, n\}$
- groups[11] — the functions $ax+b$ with $a \neq 0$, a second non-abelian group under composition
- proof-techniques[12] — contraposition and counterexample
- 2026-09-08-composition-of-functions[13] — Lecture 20 · 2026-09-07-functions-one-to-one-onto-inverse[14] — Lecture 19

[7] courses/math1061/2026-09-08-composition-of-functions.pdf
[8] courses/math1061/one-to-one-and-onto-functions.pdf
[9] courses/math1061/inverse-functions.pdf
[10] courses/math1061/symmetric-group.pdf
[11] courses/math1061/groups.pdf
[12] courses/math1061/proof-techniques.pdf
[13] courses/math1061/2026-09-08-composition-of-functions.pdf
[14] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf

- practice-problems[15] — §26

[15] courses/math1061/practice-problems.pdf