Antisymmetric Relations

Table of contents

The definition . . . 1
In the arrow diagram . . . 2
Antisymmetric is not "not symmetric" . . . 3
Proving and disproving . . . 4
Examples . . . 5
Common mistakes . . . 6
See also . . . 6

A relation is antisymmetric when no two *different* elements are related in both directions. It is the property that takes the place of symmetry in a partial order (partial-order-relations[1]). In spite of the name it is not the opposite of symmetric. The other three properties are in reflexive-symmetric-and-transitive-relations[2]. First met in 2026-09-22-partial-order-relations[3].

The definition

A relation R on a set S is **antisymmetric** if and only if

$$\forall a, b \in S,\ (a\ R\ b \text{ and } b\ R\ a) \to (a = b).$$

There are two **equivalent definitions**:

$$\forall a, b \in S,\ (a \neq b) \to \sim(a\ R\ b \text{ and } b\ R\ a).$$
$$\forall a, b \in S,\ (a\ R\ b \text{ and } a \neq b) \to \sim(b\ R\ a).$$

[1] courses/math1061/partial-order-relations.pdf
[2] courses/math1061/reflexive-symmetric-and-transitive-relations.pdf
[3] courses/math1061/2026-09-22-partial-order-relations.pdf

The first is the **easiest for proving** a relation antisymmetric, and the third is the most **intuitive**.

	In words
first	If a and b are related both ways, they are the same element.
second	Two different elements are not related both ways.
third	If a is related to a different element b, then b is not related back to a.

Why the three say the same thing

Write p for $a\ R\ b$, q for $b\ R\ a$ and r for $a = b$. The three definitions are the conditionals

$$(p \wedge q) \to r, \qquad \sim r \to \sim(p \wedge q), \qquad (p \wedge \sim r) \to \sim q.$$

The second is the contrapositive of the first (conditional-statements[4]). For the third, recall that a conditional is false in only one situation, a true hypothesis with a false conclusion. Work out that situation for each:

Conditional	False exactly when
$(p \wedge q) \to r$	p true, q true, r false
$\sim r \to \sim(p \wedge q)$	r false, p true, q true
$(p \wedge \sim r) \to \sim q$	p true, r false, q true

It is the same situation each time, so the three are logically equivalent. Each one forbids exactly one thing: $a\ R\ b$ **and** $b\ R\ a$ **with** $a \neq b$.

In the arrow diagram

Antisymmetric: no two different dots are joined by arrows both ways. Loops are not restricted. When $a = b$ the first definition reads ($a\ R\ a$ and $a\ R\ a$) $\to$ ($a = a$), which is true whether or not there is a loop at a.

Between two different elements there are three possibilities, and symmetry and antisymmetry each rule out a different one.

[4] courses/math1061/conditional-statements.pdf

between two different elements		symmetric	antisymmetric
	no arrow	✓ allowed	✓ allowed
	one arrow	× no return	✓ allowed
	arrows both ways	✓ allowed	× but $a \neq b$
	a loop	✓ allowed	✓ allowed

Figure 1: What symmetry and antisymmetry allow between two different elements. With no arrow, both are satisfied. A single arrow has no return, which breaks symmetry and is fine for antisymmetry. Arrows both ways are what symmetry asks for, and they break antisymmetry. So a symmetric relation has no single arrows, and an antisymmetric relation has no two-way pairs. A loop joins an element to itself and breaks neither.

Antisymmetric is not "not symmetric"

Antisymmetric $\neq$ not symmetric. The two properties are independent, and all four combinations occur. The diagrams are the four of 2026-09-22-partial-order-relations[5].

	antisymmetric	**not antisymmetric**
symmetric	Diagram 2. Only loops. Also: $=$ on any set.	Diagram 3. Two-way pairs, no single arrows. Also: same parity on $\mathbb{Z}$.
not symmetric	Diagram 4. Single arrows, no two-way pairs. Also: $\leq$ on $\mathbb{R}$, $\subseteq$, divides on $\mathbb{Z}^+$.	Diagram 1. A single arrow and a two-way pair.

The definitions make the difference exact:

- **not symmetric** says *there is* an arrow with no return;
- **antisymmetric** says *no* arrow between different elements has a return.

[5] courses/math1061/2026-09-22-partial-order-relations.pdf

A relation that is not symmetric can still have some two-way pairs, as Diagram 1 does, and then it is not antisymmetric either.

Both at once. If R is symmetric and antisymmetric, every arrow is a loop. Suppose $a \mathrel{R} b$. Symmetry gives $b \mathrel{R} a$, and then antisymmetry gives $a = b$. So the only relations with both properties are those in which no element is related to anything but itself, like Diagram 2.

Proving and disproving

Antisymmetry is a "for all", so it is proved with arbitrary elements and disproved with one witness.

	How
To prove it	Let $a, b \in S$ and suppose $a \mathrel{R} b$ **and** $b \mathrel{R} a$. Unfold both. Show $a = b$.
To disprove it	Find two **different** elements a, b with $a \mathrel{R} b$ and $b \mathrel{R} a$.

In the proof, write the goal down before starting: *(Goal: show $a = b$).* Both hypotheses will be needed.

Worked: divides on $\mathbb{Z}^+$

Worked in full in 2026-09-22-partial-order-relations[6]. Let $a, b \in \mathbb{Z}^+$ and suppose $a \mid b$ and $b \mid a$. *(Goal: show $a = b$.)* Then $b = ak$ and $a = b\ell$ for some $k, \ell \in \mathbb{Z}$. Substituting,

$$b = ak = (b\ell)k = b(\ell k),$$

and since $b \neq 0$, $\ell k = 1$. Now $a, b > 0$ forces $k > 0$ and $\ell > 0$, and the only positive integers with product 1 are $\ell = k = 1$. So $a = b$.

The last step uses that the set is $\mathbb{Z}^+$. See Examples for what goes wrong on $\mathbb{Z}$.

Worked: a disproof

From practice-problems[7], Q2. On binary strings of length 6, let $s \mathrel{\rho} t$ if and only if weight$(s) \leq$ weight(t), where the weight is the number of ones. Take $s = 100000$ and $t = 010000$. Both have weight 1, so $s \mathrel{\rho} t$ and $t \mathrel{\rho} s$, but $s \neq t$. So ρ is not antisymmetric.

Two different things with the same weight are related both ways. Any relation that compares elements through a number they can share fails in the same way.

[6] courses/math1061/2026-09-22-partial-order-relations.pdf
[7] courses/math1061/practice-problems.pdf

The third form in a proof

The first definition is the easiest to prove from, but the third works too. The solution to practice-problems[8], Q1 uses it: *suppose* $x \rho y$ *where* $x \neq y$*; we must show that then* $y \not\rho x$. Either is acceptable. State which you are using.

Examples

The standard examples, with reasons:

Relation	on	Antisymmetric because
$\leq$	$\mathbb{R}$	$(x \leq y$ and $y \leq x) \to x = y$
$\geq$	$\mathbb{R}$	the same, reversed
$<$, $>$	$\mathbb{R}$	$x < y$ and $y < x$ never hold together
divides	$\mathbb{Z}^+$	proved above
$\subseteq$	any set S of sets	$(X \subseteq Y$ and $Y \subseteq X) \to X = Y$

$<$ is antisymmetric vacuously. For no x, y are $x < y$ and $y < x$ both true. The hypothesis of the first definition is therefore always false, and a conditional with a false hypothesis is true (conditional-statements[9]). In the diagram of $<$ there are no two-way pairs, and no loops either.

$\subseteq$ is the double-containment rule. Showing $X \subseteq Y$ and $Y \subseteq X$ to conclude $X = Y$ is how set equality is proved (sets-and-set-notation[10]). Antisymmetry of $\subseteq$ is that rule under another name.

Divides is not antisymmetric on $\mathbb{Z}$. $2 \mid -2$, since $-2 = 2 \cdot (-1)$, and $-2 \mid 2$, since $2 = (-2)(-1)$, but $2 \neq -2$. This is why the statement is about $\mathbb{Z}^+$. A relation is always a relation on a particular set, and the same rule on a bigger set can lose a property.

Not antisymmetric: any relation with two different elements related both ways. That includes congruence mod n for $n \geq 1$, same parity, and every equivalence relation that has a class with more than one element.

[8] courses/math1061/practice-problems.pdf
[9] courses/math1061/conditional-statements.pdf
[10] courses/math1061/sets-and-set-notation.pdf

Common mistakes

- **Reading "antisymmetric" as "not symmetric".** See the four combinations above.
- **Thinking a loop breaks antisymmetry.** It does not. The property is about pairs of *different* elements.
- **Proving it from an example.** "$2 \mid 4$ but $4 \nmid 2$" shows one pair behaves. Antisymmetry is about all pairs.
- **Using only one hypothesis.** A proof that starts "suppose $a \; R \; b$" and never uses $b \; R \; a$ cannot reach $a = b$.
- **Disproving with $a = b$.** A witness needs two different elements. $a \; R \; a$ on its own shows nothing.

See also

- partial-order-relations[11] — reflexive, antisymmetric and transitive together
- reflexive-symmetric-and-transitive-relations[12] — the other three properties, and how to prove and disprove them
- conditional-statements[13] — the contrapositive, and vacuous truth
- divisibility-and-factorisation[14] — the definition of divides
- 2026-09-22-partial-order-relations[15] — Lecture 25
- practice-problems[16] — §32, with full worked solutions

[11] courses/math1061/partial-order-relations.pdf
[12] courses/math1061/reflexive-symmetric-and-transitive-relations.pdf
[13] courses/math1061/conditional-statements.pdf
[14] courses/math1061/divisibility-and-factorisation.pdf
[15] courses/math1061/2026-09-22-partial-order-relations.pdf
[16] courses/math1061/practice-problems.pdf