Lecture 35 — Matrix Representations of Graphs

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Two ways to write a graph down as a table of numbers. The **adjacency matrix** has a row and a column for each vertex and records how many edges join each pair of vertices. The **incidence matrix** has a row for each vertex and a column for each edge and records how many times each vertex is incident with each edge. Both go in both directions: a graph gives a matrix, and a matrix of the right kind gives a graph. Pre-work: Video 045 (Matrix Representations of Graphs). The definitions, how loops are counted, reading a graph off a matrix and which matrices qualify are in adjacency-matrices[1] and incidence-matrices[2]. Graphs, degree and the Handshake Theorem are in graphs[3].

[1] courses/math1061/adjacency-matrices.pdf
[2] courses/math1061/incidence-matrices.pdf
[3] courses/math1061/graphs.pdf

Goals

1. Learn how to represent a graph by an adjacency matrix.
2. Learn how to represent a graph by an incidence matrix.
3. Learn how to draw a graph from its adjacency matrix.
4. Learn how to draw a graph from its incidence matrix.

Two questions to keep in mind:

- **In an incidence matrix, which are the rows and which are the columns: the vertices or the edges?** The vertices are the rows and the edges are the columns, as in the definition. There is a way to remember it in which way round.
- **Is it OK to swap them?** The swapped matrix holds the same information, but it is not the incidence matrix as this course defines it. Also in which way round.

The running example

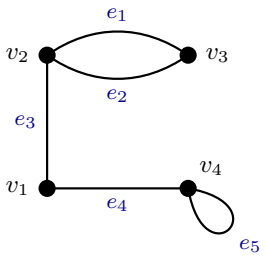
One graph is used all the way through. It has vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{e_1, e_2, e_3, e_4, e_5\}$, with

Edge	Endpoints
e_1	$\{v_2, v_3\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_1, v_2\}$
e_4	$\{v_1, v_4\}$
e_5	$\{v_4\}$

So e_1 and e_2 are parallel edges, and e_5 is a loop at v_4. The degrees are $\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 2$ and $\deg(v_4) = 3$, the loop adding 2 to $\deg(v_4)$. They add to $10 = 2 \cdot 5$.

Worked: edges between pairs of vertices

For the running example, how many edges have the following as endpoints?
(a) $\{v_1, v_2\}$
(b) $\{v_1, v_3\}$
(c) $\{v_1, v_4\}$
(d) $\{v_2, v_3\}$
(e) $\{v_4\}$



$A =$

	v_1	v_2	v_3	v_4
v_1	0	1	0	1
v_2	1	0	**2**	0
v_3	0	**2**	0	0
v_4	1	0	0	**1**

$N =$

	e_1	e_2	e_3	e_4	e_5
v_1	0	0	1	1	0
v_2	1	1	1	0	0
v_3	1	1	0	0	0
v_4	0	0	0	1	**2**

Figure 1: The running example and its two matrices. The adjacency matrix A has a row and a column for each vertex, and each entry counts the edges joining that pair of vertices: the 2 is the pair of parallel edges between v2 and v3, and the 1 on the diagonal is the single loop at v4. The incidence matrix N has a row for each vertex and a column for each edge, and each entry counts how many times that vertex is incident with that edge: the 2 is the loop, which meets v4 at both of its ends.

	Endpoints	Edges	Number
(a)	$\{v_1, v_2\}$	e_3	1
(b)	$\{v_1, v_3\}$	none	0
(c)	$\{v_1, v_4\}$	e_4	1
(d)	$\{v_2, v_3\}$	e_1, e_2	2
(e)	$\{v_4\}$	e_5	1

For (e), the set of endpoints $\{v_4\}$ has one element, so the edges with that set of endpoints are the loops at v_4. There is one, e_5.

These five numbers are entries of the adjacency matrix. Answers (a), (b) and (c), with a 0 for "no loop at v_1", are the first row, 0 1 0 1. Answer (d) is the entry in row v_2, column v_3, and answer (e) is the diagonal entry in row v_4, column v_4.

The adjacency matrix

> The **adjacency matrix** of a graph with vertices $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix $A = [a_{ij}]$ where each entry a_{ij} is the number of edges with endpoints $\{v_i, v_j\}$.

When $i = j$ the set $\{v_i, v_i\}$ is just $\{v_i\}$, so a diagonal entry a_{ii} is the **number of loops** at v_i. A loop is one edge, so it counts **1** on the diagonal. In the running example $a_{44} = 1$ for the single loop e_5.

Three things follow straight from the definition:

- $\{v_i, v_j\} = \{v_j, v_i\}$, so $a_{ij} = a_{ji}$: **the matrix is symmetric**.
- Every entry counts edges, so every entry is a **non-negative integer**.
- A loop adds 2 to a degree but only 1 to the matrix. So $\deg(v_i)$ **is the row sum plus the diagonal entry**. In the running example row v_4 adds to $1 + 0 + 0 + 1 = 2$, and $\deg(v_4) = 2 + a_{44} = 3$.

More on all three, and the proofs, in adjacency-matrices[4].

Worked: write down the adjacency matrix

Write down the adjacency matrix for the graph below.

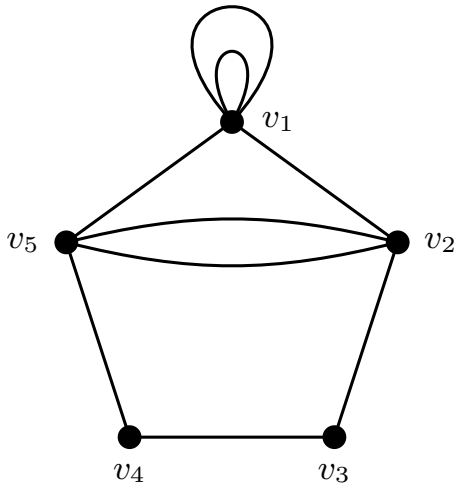


Figure 2: The graph for the first question: five vertices around a pentagon, two loops at v1, and two parallel edges between v5 and v2. It has 9 edges in all.

Answer.

$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 2 & 1 & 0 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 2 \\ v_3 & 0 & 1 & 0 & 1 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 2 & 0 & 1 & 0 \end{array}$$

[4] courses/math1061/adjacency-matrices.pdf

Go through the pairs once each, then copy across the diagonal, since the matrix is symmetric.

- **Diagonal.** Two loops at v_1, so $a_{11} = 2$. No other vertex has a loop.
- **Row v_1.** One edge to v_2, one to v_5, none to v_3 or v_4.
- **Row v_2.** One edge to v_3 and two parallel edges to v_5, so $a_{25} = 2$.
- **Rows v_3 and v_4.** One edge v_3v_4 and one edge v_4v_5.

Two checks.

- **Edge count.** Each edge is counted once in the upper triangle with the diagonal: $(2+1+0+0+1)+(0+1+0+2)+(0+1+0)+(0+1)+0 = 4+3+1+1+0 = 9$, the 9 edges of the graph.
- **Degrees.** Row sum plus diagonal entry: $\deg(v_1) = 4+2 = 6$, the two loops giving 4 of it, and $\deg(v_2) = 4$, $\deg(v_3) = 2$, $\deg(v_4) = 2$, $\deg(v_5) = 4$. They add to $18 = 2 \cdot 9$, as the graphs[5] says they must.

Worked: draw the graph of an adjacency matrix

Draw the graph with this adjacency matrix.

	v_1	v_2	v_3	v_4	v_5
v_1	2	1	0	3	1
v_2	1	0	0	0	2
v_3	0	0	0	1	0
v_4	3	0	1	0	1
v_5	1	2	0	1	0

First check that it can be an adjacency matrix: it is square, symmetric and its entries are non-negative integers. Then read **only the diagonal and the entries above it**. Each entry below the diagonal repeats one above it, and reading both would draw every edge twice.

Entry	Value	Edges
a_{11}	2	two loops at v_1
a_{12}	1	one edge v_1v_2
a_{14}	3	three parallel edges v_1v_4
a_{15}	1	one edge v_1v_5
a_{25}	2	two parallel edges v_2v_5
a_{34}	1	one edge v_3v_4
a_{45}	1	one edge v_4v_5

Every other entry on or above the diagonal is 0. That is $2+1+3+1+2+1+1 = 11$ edges.

[5] courses/math1061/graphs.pdf

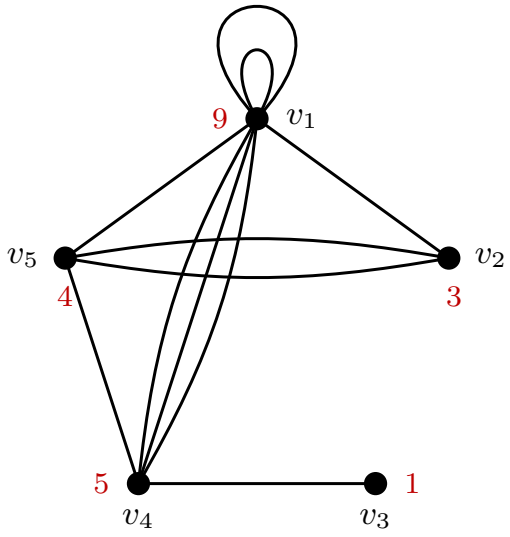


Figure 3: The graph drawn from the matrix: two loops at v1, three parallel edges between v1 and v4, two between v2 and v5, and single edges v1v2, v1v5, v3v4 and v4v5. The three v1v4 edges cross the two v2v5 edges, but crossings are not vertices. The degrees, in red, are each row sum plus the diagonal entry, so v1 has degree $7 + 2 = 9$. They add to 22, twice the 11 edges.

The degrees are $\deg(v_1) = 7 + 2 = 9$, $\deg(v_2) = 3$, $\deg(v_3) = 1$, $\deg(v_4) = 5$ and $\deg(v_5) = 4$, a total of $22 = 2 \cdot 11$. Where the vertices go on the page is free. The matrix fixes only which vertices are joined, and how many times.

Worked: which is an adjacency matrix

Which of the following is an adjacency matrix for a graph?

A. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 73 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Answer: C.

A matrix is an adjacency matrix of some graph exactly when it is **square**, **symmetric** and has **non-negative integer** entries (adjacency-matrices[6]).

- **A** is square and symmetric, but -1 cannot be a number of edges.
- **B** is 2×3. An adjacency matrix has one row and one column for each vertex, so it is always square.
- **C** is 1×1, trivially symmetric, and 73 is a non-negative integer. It is the graph with **one vertex and 73 loops**. That vertex has degree $73 + 73 = 146$, and the graph has 73 edges.
- **D** is square with non-negative integer entries, but not symmetric: $a_{24} = 2$ and $a_{42} = 1$. The number of edges between v_2 and v_4 cannot be both 2 and 1. Every other pair matches across the diagonal; this one pair is enough.

Worked: how many times is a vertex incident with an edge

For the running example:
(a) How many times is v_1 incident with e_1?
(b) How many times is v_1 incident with e_4?
(c) How many times is v_2 incident with e_2?
(d) How many times is v_4 incident with e_5?

[6] courses/math1061/adjacency-matrices.pdf

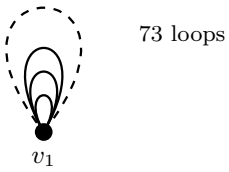


Figure 4: The graph of the 1 by 1 matrix [73]: a single vertex with 73 loops, three of them drawn and the dashed one standing for the rest. Its one diagonal entry counts the loops, 73, and its degree is twice that, 146.

(a) 0. The endpoints of e_1 are v_2 and v_3, so e_1 does not meet v_1.

(b) 1. e_4 joins v_1 and v_4. One of its ends is at v_1.

(c) 1. e_2 joins v_2 and v_3. One of its ends is at v_2.

(d) 2. e_5 is a loop at v_4. Both of its ends are at v_4, so it meets v_4 twice. This is the same count that makes a loop add 2 to a degree (graphs[7]).

The incidence matrix

> The **incidence matrix** of a graph with vertices $v_1, v_2, \dots, v_n$ and edges $e_1, e_2, \dots, e_m$ is the $n \times m$ matrix $N = [n_{ij}]$ where each entry n_{ij} is the number of times vertex v_i is incident with edge e_j.

The answers above are four of its entries: $n_{11} = 0$, $n_{14} = 1$, $n_{22} = 1$ and $n_{45} = 2$. The whole matrix for the running example is in the figure.

- **A column is an edge.** It has a 1 at each of the edge's two endpoints, or a single 2 at the vertex of a loop. Either way **every column adds to 2**: every edge has two ends.
- **A row is a vertex.** It adds up the ends of edges at that vertex, so **the row sum is the degree**, loops included. In the running example the rows add to $2, 3, 2, 3$.
- **Parallel edges are identical columns.** The columns of e_1 and e_2 are both $(0, 1, 1, 0)$.

Adding up all the entries by columns gives $2m$, and by rows gives the sum of the degrees. That is the graphs[8] again. More in incidence-matrices[9].

[7] courses/math1061/graphs.pdf
[8] courses/math1061/graphs.pdf
[9] courses/math1061/incidence-matrices.pdf

Which way round

Rows are vertices, columns are edges. One way to remember it: the entry n_{ij} answers "how many times is **vertex** v_i incident with **edge** e_j", and the index that comes first, i, is the row, so the vertex comes first. Another: the adjacency matrix has a row for each vertex, and so does the incidence matrix. Both matrices have n rows, one per vertex; they differ only in their columns, vertices for A and edges for N.

Swapping them gives the transpose N^{T}, the $m \times n$ matrix with a row for each edge. It holds exactly the same information. The graph can be rebuilt from it just as well, with every *row* adding to 2 and every *column* adding to a degree. But it is not the incidence matrix as this course defines it. When asked for the incidence matrix, put the vertices down the side and the edges across the top, and label both.

The question does not arise for the adjacency matrix, because it is symmetric: swapping rows and columns gives the same matrix. What does matter for both is the **order** of the vertices, and for N the order of the edges. A different order permutes the rows or columns and gives a different matrix for the same graph, which is why the rows and columns should always be labelled.

Worked: write down the incidence matrix

Write down the incidence matrix for the graph below.

Answer.

$$N = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline v_1 & 2 & 0 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 \\ v_5 & 0 & 1 & 1 & 1 \end{array}$$

Fill it one column, one edge, at a time.

- e_1 is a loop at v_1: a 2 in row v_1 and 0 elsewhere.
- e_2 and e_3 both join v_5 and v_2: a 1 in rows v_2 and v_5. Being parallel, they give the same column.
- e_4 joins v_5 and v_4: a 1 in rows v_4 and v_5.

Checks. Every column adds to 2. The row sums are the degrees, $2, 2, 0, 1, 3$, adding to $8 = 2 \cdot 4$. The row of v_3 is all zeros, because v_3 is isolated: a vertex still gets its row even when no edge meets it.

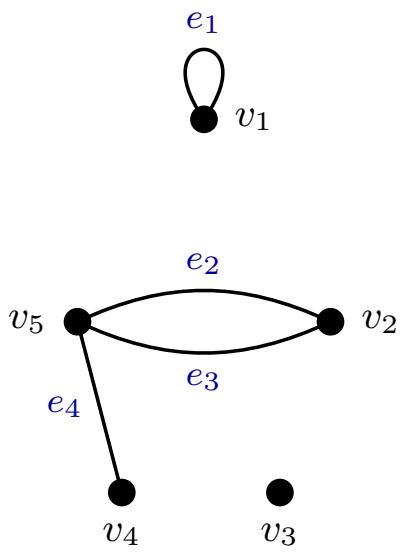


Figure 5: The graph for the incidence-matrix question: a loop e1 at v1, two parallel edges e2 and e3 between v5 and v2, an edge e4 from v5 to v4, and an isolated vertex v3. The loop gives v1 the entry 2 in column e1, and v3, which meets no edge, gives a row of zeros.

Worked: draw the graph of an incidence matrix

Draw the graph with this incidence matrix.

	e_1	e_2	e_3	e_4
v_1	2	1	0	0
v_2	0	1	0	0
v_3	0	0	2	0
v_4	0	0	0	1
v_5	0	0	0	1

Every column adds to 2, so it is an incidence matrix. Read it one column at a time. A column with two 1s is an edge between those two vertices, and a column with a single 2 is a loop.

Column	Entries	Edge
e_1	2 at v_1	loop at v_1
e_2	1 at v_1, 1 at v_2	v_1v_2
e_3	2 at v_3	loop at v_3
e_4	1 at v_4, 1 at v_5	v_4v_5

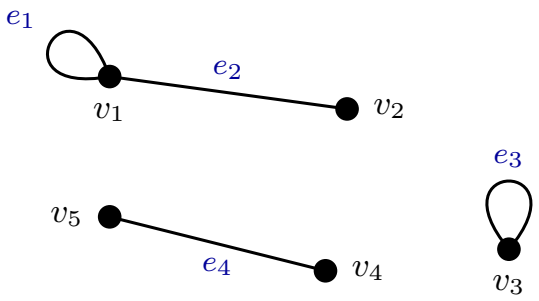


Figure 6: The graph drawn from the incidence matrix: each column is one edge. Columns e1 and e3 each have a single 2, so they are loops, at v1 and at v3. Columns e2 and e4 each have two 1s, so they join v1 to v2 and v4 to v5. The graph falls into three pieces. The row sums 3, 1, 2, 1, 1 are the degrees, and add to 8, twice the 4 edges.

Worked: which is an incidence matrix

Which of the following is an incidence matrix for a graph?

A. $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Answer: B.

A matrix is an incidence matrix of some graph exactly when its entries are non-negative integers and **every column adds to 2**, so that each column is either two 1s or a single 2 (incidence-matrices[10]). Check the columns.

	Column sums	Verdict
A	1, 3	no: the first column is an edge with one end, and no vertex can meet an edge 3 times
B	2, 2, 2	**yes**
C	1, 1	no: each column is an edge with only one end
D	2, 4, 2, 4	no: columns 2 and 4 would be edges with four ends

B has two rows and three columns, so two vertices and three edges, and each column $(1, 1)$ is an edge joining the two vertices. It is **two vertices joined by three parallel edges**.

In **C** a column with a single 1 is not a loop: a loop meets its vertex twice and is a single 2. In **D** the first and third columns are fine on their own, an edge v_2v_4 and a loop at v_2, but one bad column is enough to rule a matrix out. A square shape, or symmetry, has nothing to do with it: B is not square, and C is square and symmetric.

[10] courses/math1061/incidence-matrices.pdf

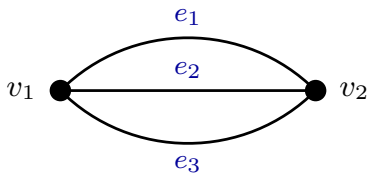


Figure 7: The graph of matrix B: two vertices and three edges. Each column of B is (1, 1), a 1 at each vertex, so each edge joins v1 and v2. The three columns are identical, which is exactly what three parallel edges look like in an incidence matrix.

Worked: Eulerian from the incidence matrix

For each of the following incidence matrices, determine whether the associated graph G is Eulerian. Explain why or why not.

(a) $N =$

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	1	0	1	1	0	1
v_3	0	1	1	0	1	1
v_4	0	0	0	1	1	0

(b) $N =$

	a	b	c	d
u	1	0	1	0
v	0	1	0	1
w	0	1	0	1
x	1	0	1	0

A graph is **Eulerian** if it has an Euler circuit, a circuit that contains every edge (euler-circuits-and-trails[11]). The test is the theorem (euler-circuits-and-trails[12]):

> A **connected** graph has an Euler circuit if and only if every vertex has even degree.

The incidence matrix gives both halves of the test. The degrees are the row sums, and the columns say which vertices each edge joins, which settles whether the graph is connected.

(a) Yes, G is Eulerian.

The row sums give the degrees:

$$\deg(v_1) = 2, \quad \deg(v_2) = 4, \quad \deg(v_3) = 4, \quad \deg(v_4) = 2.$$

All four are even. Reading the columns, the edges are $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_2v_3$, $e_4 = v_2v_4$, $e_5 = v_3v_4$ and $e_6 = v_2v_3$, so e_3 and e_6 are parallel. G is connected: v_2 is joined to each of v_1, v_3 and v_4. A connected graph with every degree even has an Euler circuit, for example

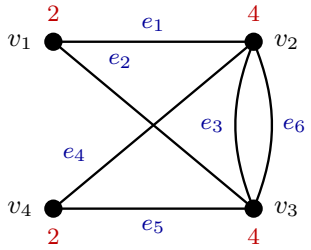
[11] courses/math1061/euler-circuits-and-trails.pdf
[12] courses/math1061/euler-circuits-and-trails.pdf

$$v_1\ e_1\ v_2\ e_3\ v_3\ e_6\ v_2\ e_4\ v_4\ e_5\ v_3\ e_2\ v_1.$$

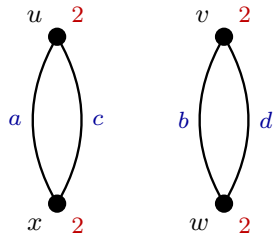
It uses $e_1, e_3, e_6, e_4, e_5, e_2$, each of the six edges exactly once, each edge joins the vertices on either side of it in the list, and it ends where it started.

(b) No, G is not Eulerian.

Every row adds to 2, so every vertex has even degree. But the columns show the edges are $a = ux$, $b = vw$, $c = ux$ and $d = vw$. No edge joins u or x to v or w, so G is **not connected**: it is two separate pieces, u and x joined by a and c, and v and w joined by b and d. A circuit is a walk, and a walk only moves along edges, so it never leaves the piece it starts in. No circuit can contain both a and b, so there is no Euler circuit. Even degrees are not enough on their own: the theorem needs the graph to be connected.



(a) connected, all even: Eulerian



(b) two pieces: not Eulerian

Figure 8: The two graphs read off the incidence matrices, degrees in red. Left, (a): every degree is even and the graph is connected, so it has an Euler circuit; the diagonals e2 and e4 cross without meeting. Right, (b): every degree is 2, but the graph is two separate pieces, and a circuit cannot get from one piece to the other, so no circuit uses every edge.

See also

- adjacency-matrices[13] — the definition, loops on the diagonal, reading a graph off, and which matrices are adjacency matrices
- incidence-matrices[14] — the definition, column sums and row sums, reading a graph off, which matrices qualify, and how the two matrices compare
- graphs[15] — loops, parallel edges, degree and the Handshake Theorem

[13] courses/math1061/adjacency-matrices.pdf
[14] courses/math1061/incidence-matrices.pdf
[15] courses/math1061/graphs.pdf

- euler-circuits-and-trails[16] — Euler circuits and trails and the even-degree theorem
- walks-trails-and-circuits[17] — walks, circuits and connected graphs
- 2026-10-26-walks-trails-and-circuits[18] — Lecture 34
- 2026-10-29-trees[19] — Lecture 36
- practice-problems[20] — §45, with full worked solutions

[16] courses/math1061/euler-circuits-and-trails.pdf
[17] courses/math1061/walks-trails-and-circuits.pdf
[18] courses/math1061/2026-10-26-walks-trails-and-circuits.pdf
[19] courses/math1061/2026-10-29-trees.pdf
[20] courses/math1061/practice-problems.pdf