Lecture 20 — Composition of Functions

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Pre-work: Video 026 (Composition of functions). The reference note is composition-of-functions[1]; the properties it builds on are in one-to-one-and-onto-functions[2] and inverse-functions[3]. The order here: the one-to-one and onto templates again, two worked examples on inverses, then the composition results.

> **Date.** On the schedule this is the Thursday 10/09 lecture; on the one-ahead reading explained under math1061[4] it is Tuesday 08/09.

Learning goals

- Review definitions of injective and surjective functions.
- Investigate conditions for a **composition** of functions to be one-to-one, onto.

The picture to keep in mind has three sets X, Y, Z. f sends $x \mapsto f(x)$, a hop $X \to Y$. g sends $f(x) \mapsto g(f(x))$, a hop $Y \to Z$. $g \circ f$ goes from x straight to $g(f(x))$.

[1] courses/math1061/composition-of-functions.pdf
[2] courses/math1061/one-to-one-and-onto-functions.pdf
[3] courses/math1061/inverse-functions.pdf
[4] courses/math1061/math1061.pdf

Worked: is $f \circ g$ also a bijection?

If $f : X \to Y$ and $g : Y \to Z$ are both bijective functions, then the composition $g \circ f : X \to Z$ is also a bijection. In this case, is $f \circ g$ also a bijection?

For $f \circ g$ to be defined, we need $X = Z$. With that, yes, $f \circ g$ is a bijection.

$f \circ g$ means *g first, then f*: $Y \xrightarrow{g} Z \xrightarrow{f} Y$, with $Z = X$. So g's outputs — elements of Z — have to be things f accepts — elements of X. Unless $Z = X$ the expression is not meaningless-but-false; it is simply undefined. **Composition is not commutative, and most of the time the other order does not even exist.**

The one-to-one and onto templates

The four templates were first met in 2026-09-07-functions-one-to-one-onto-inverse[5]. Going *backwards* along f shows the two ways an inverse fails. When f is not one-to-one, the shared value $f(x_1) = f(x_2)$ leads back to two sources. When f is not onto, the unreached element of Y has nothing to lead back to.

Injective (one-to-one): to show, suppose $f(x_1) = f(x_2)$ and show $x_1 = x_2$; to show *not*, find $x_1 \neq x_2$ with $f(x_1) = f(x_2)$.

Surjective (onto): to show, suppose $y \in Y$ and find $x \in X$ with $f(x) = y$; to show *not*, find $y \in Y$ with $f(x) \neq y$ for all x.

Bijective: both. A bijection $f : X \to Y$ has an inverse function $f^{-1} : Y \to X$, with $f^{-1}(y) = x$ where x is the unique element in X such that $f(x) = y$. The converse holds too, which makes it an equivalence:

$$f \text{ has an inverse} \iff f \text{ is a bijection.}$$

Worked: the largest prime divisor

> Consider the function $f : \mathbb{Z} - \{-1, 0, 1\} \to \mathbb{Z}^{>1}$ defined by $f(x) = p$, where p is the largest prime divisor of x.

Some values first, to see what the function does: $f(2) = 2$, $f(10) = 5$, $f(-6) = 3$. Removing -1, 0, 1 from the domain is what makes it a function at all: ± 1 have no prime divisor, and 0 is divisible by *every* prime, so it has no largest one.

(a) Is f an injection? $f(10) = 5$, $f(15) = 5$ and $10 \neq 15$. **Not injective.**

[5] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf

(b) Is f a surjection? No. $4 \in \mathbb{Z}^{>1}$ but $4 = 2 \cdot 2$ is not prime, so $\nexists\, x \in \mathbb{Z} - \{-1, 0, 1\}$ s.t. $f(x) = 4$.

(c) Is f a bijection? If it exists, determine f^{-1}. No. f^{-1} does not exist (*d.n.e.*).

The two disproofs are both single witnesses, as the templates say they should be. For (b) the witness is any composite number — the range of f is the primes, and $\mathbb{Z}^{>1}$ has far more in it.

Worked: reading f^{-1} off the onto proof

> The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f((x, y)) = (y + 1, 2x)$ is both one-to-one and onto (proved in 2026-09-07-functions-one-to-one-onto-inverse[6]). **Determine f^{-1}.**

The onto half of that proof is the part to use. The candidate it named was $\left(\frac{b}{2}, a - 1\right)$. That candidate **is** the inverse:

$$f^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \qquad f^{-1}((a, b)) = \left(\tfrac{b}{2},\ a - 1\right),$$

or, renaming the variables to the usual ones, $f^{-1}((x, y)) = (\frac{y}{2},\ x - 1)$.

Sense check: $f \circ f^{-1}((x, y)) = (x, y)$ and $f^{-1} \circ f((x, y)) = (x, y)$. Run one of them:

$$f(f^{-1}((x, y))) = f((\tfrac{y}{2}, x - 1)) = ((x - 1) + 1,\ 2 \cdot \tfrac{y}{2}) = (x, y). \checkmark$$

The method is the one to remember: **the onto proof already computes the inverse.** Showing f is onto means, for an arbitrary (a, b), finding the (x, y) that maps to it — and the formula for that (x, y) in terms of (a, b) *is* f^{-1}. The formula is a function in its own right; one-to-one is what makes it the *inverse* — it guarantees the candidate is the only preimage, so that $f^{-1} \circ f$ is the identity as well as $f \circ f^{-1}$. There is no separate "find the inverse" technique.

Composition of functions

> Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the **composition** of f and g is the function $g \circ f : X \to Z$ defined by the rule
>
> $$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X.$$

Read $g \circ f$ right to left: f acts first, then g. The name is in the "wrong" order because the formula $g(f(x))$ is.

Two properties pass from f and g to the composite:

[6] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf

- f and g both onto $\rightarrow$ $g \circ f$ is onto
- f and g both one-to-one $\rightarrow$ $g \circ f$ is one-to-one

$g \circ f$ has domain equal to the domain of f, which is X, and co-domain equal to the co-domain of g, which is Z. The middle set Y disappears from the type of the composite.

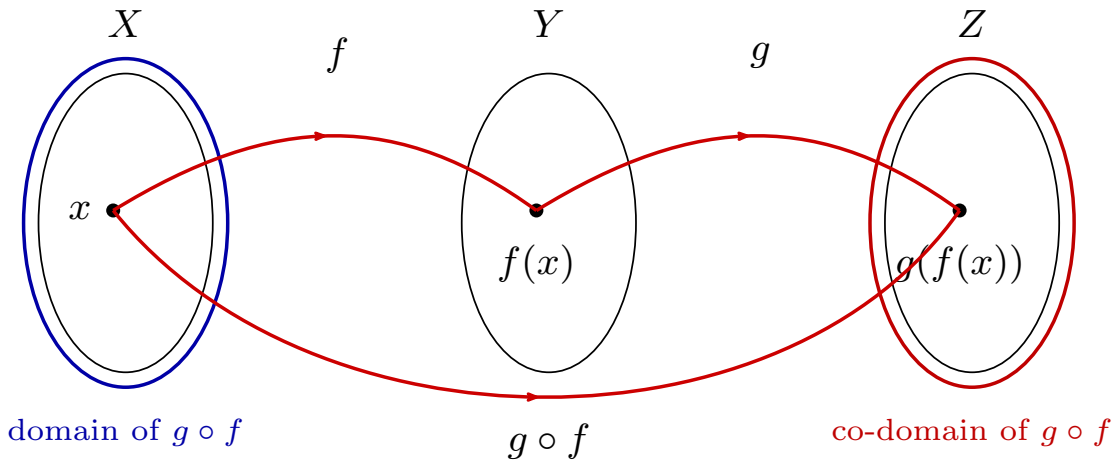


Figure 1: The composition $g \circ f$. X (blue outline) is the domain of $g \circ f$ because it is the domain of f, and Z (red outline) is its co-domain because it is the co-domain of g. The middle set Y is where f lands and g starts, and it does not appear in $g \circ f : X \rightarrow Z$ at all.

The converse: from $g \circ f$ back to f and g

Those two facts go one way. The **converse** asks which of the two ingredients must have a property when the composite has it.

For all functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$:

1. **Find a counterexample** that disproves: if $g \circ f$ is onto, then f is onto.
2. **Prove that:** if $g \circ f$ is onto, then g is onto. *(Proof by contraposition.)*
3. **Prove that:** if $g \circ f$ is one-to-one, then f is one-to-one.
4. **Find a counterexample** that disproves: if $g \circ f$ is one-to-one, then g is one-to-one.

Statements (1) and (2) have the same hypothesis, *if $g \circ f$ is onto*, and two different conclusions, one false and one true. The four results, in one table:

If $g \circ f$ is …	then f is …	then g is …
onto	not necessarily — (1)	**onto** — (2)
one-to-one	**one-to-one** — (3)	not necessarily — (4)

The property propagates to **the function on the same side as the property's quantifier**: onto is about reaching everything in Z, and it is g that lands in Z; one-to-one is about not merging anything from X, and it is f that leaves X.

(1) Counterexample

Let $X = \{a, b\}$, $Y = \{0, 1, 2\}$, $Z = \{v, w\}$. Define $f : X \to Y$ and $g : Y \to Z$ by

$$f = \{(a, 0), (b, 1)\}, \qquad g = \{(0, v), (1, w), (2, w)\}.$$

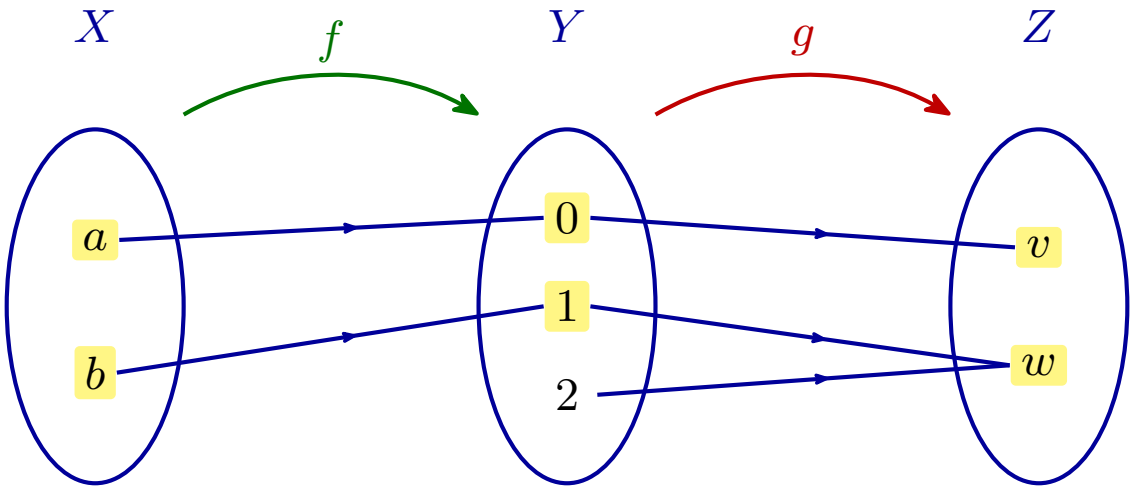
f is **not onto** (since $2 \in Y$ but $\nexists\, x \in X$ s.t. $f(x) = 2$). Then $g \circ f = \{(a, v), (b, w)\}$, which **is onto** (since $v \in Z$ and $g \circ f(a) = v$; $w \in Z$ and $g \circ f(b) = w$).

Figure 2 draws it, with yellow highlights on the elements that matter.

(2) Proof

> **We prove the contrapositive statement:** $\forall$ functions $f : X \to Y$ and $g : Y \to Z$, if g is not onto then $g \circ f$ is not onto.
>
> Let $f : X \to Y$ and $g : Y \to Z$ be functions and suppose g is not onto. Then $\exists\, z \in Z$ such that z is not in the range of g. Thus $\nexists\, x \in X$ with $g(f(x)) = z$, i.e. z is not in the range of $g \circ f$.
>
> *(why? Since if there were such an $x \in X$ we could take $y = f(x)$ to give $g(y) = z$ $(\Rightarrow\Leftarrow)$)*
>
> $\therefore g \circ f$ is not onto. $\square$

The contrapositive is the natural shape because *not onto* hands you a concrete object — a missed z — to work with, where *onto* would hand you only a hypothesis about every z. The bracketed *why?* is a tiny proof by contradiction nested inside: anything $g \circ f$ reaches, g reaches too (via $y = f(x)$), so a z that g misses, $g \circ f$ misses. Figure 3 shows it: the range of g is hatched at the top of Z, where g's arrow lands, and the lone dot z sits below it, outside the hatching.



$f = \{(a, 0), (b, 1)\}$ $\quad$ $g = \{(0, v), (1, w), (2, w)\}$

not onto: 2 is missed

$$g \circ f = \{(a, v), (b, w)\} \text{ — } \textbf{onto}$$

Figure 2: The counterexample for (1). f misses 2, so f is not onto; but g sends 1 and 2 to the same place, so the miss never shows: $g \circ f$ still reaches both v and w. The yellow highlights are on a, b, 0, 1, v, w — every element the composite actually touches — and 2 is left plain. The function g has covered for f's gap.

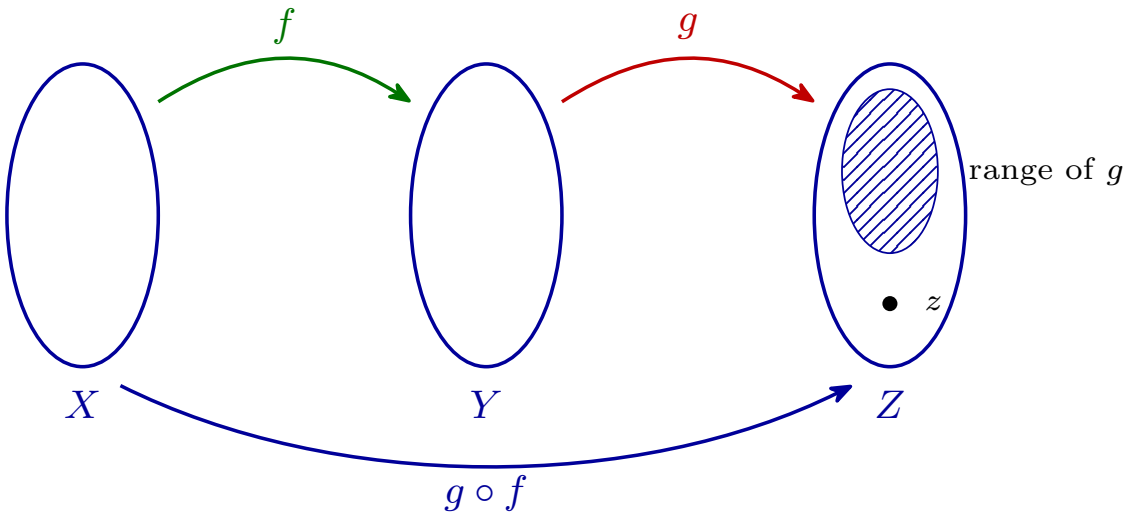


Figure 3: The proof of (2) as a picture. g's arrow lands in the hatched region, its range; the dot z sits outside it. Whatever f does, $g \circ f$ can only land where g lands, so z is outside the range of $g \circ f$ too. The mirror of the picture for (3): there a collision in f survives g, here a gap in g survives f.

(3) Proof

> **We prove the contrapositive:** $\forall$ functions $f : X \to Y$ and $g : Y \to Z$, if f is not one-to-one then $g \circ f$ is not one-to-one.
>
> Let $f : X \to Y$ and $g : Y \to Z$ be functions and suppose f is not one-to-one. Then $\exists\, x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Hence
>
> $$g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2),$$
>
> and since $x_1 \neq x_2$, $g \circ f$ is not one-to-one. $\square$

Same shape as (2), mirrored. *Not one-to-one* hands you a concrete collision $x_1 \neq x_2$, and the collision survives applying g because g is a function: equal inputs give equal outputs. Figure 4 shows the collision.

(4) Counterexample

Let $X = \{a, b\}$, $Y = \{1, 2, 3\}$, $Z = \{u, v\}$. Define $f : X \to Y$ and $g : Y \to Z$ by

$$f = \{(a, 1), (b, 2)\}, \qquad g = \{(1, u), (2, v), (3, v)\}.$$

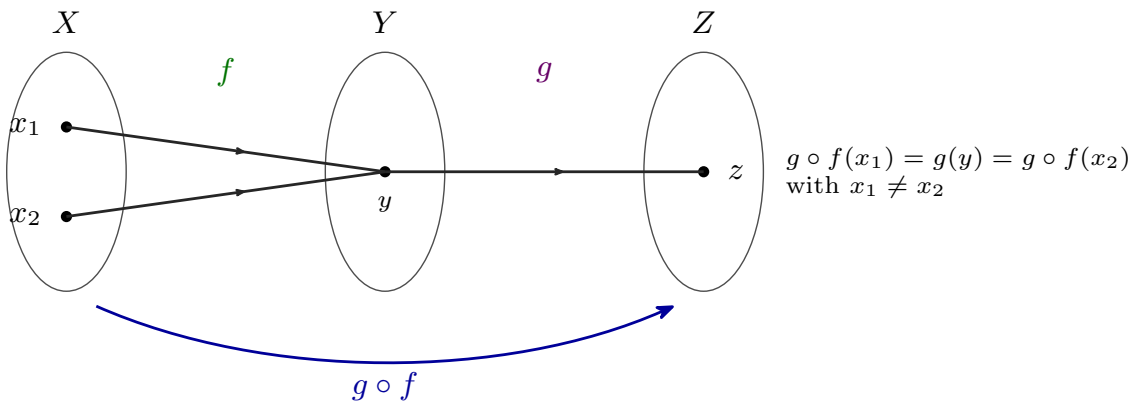


Figure 4: The proof of (3) as a picture. Two different x's collide under f at one y; g then has only one y to act on, so it can only send them both to the same z. A collision in f is a collision in $g \circ f$, whatever g does.

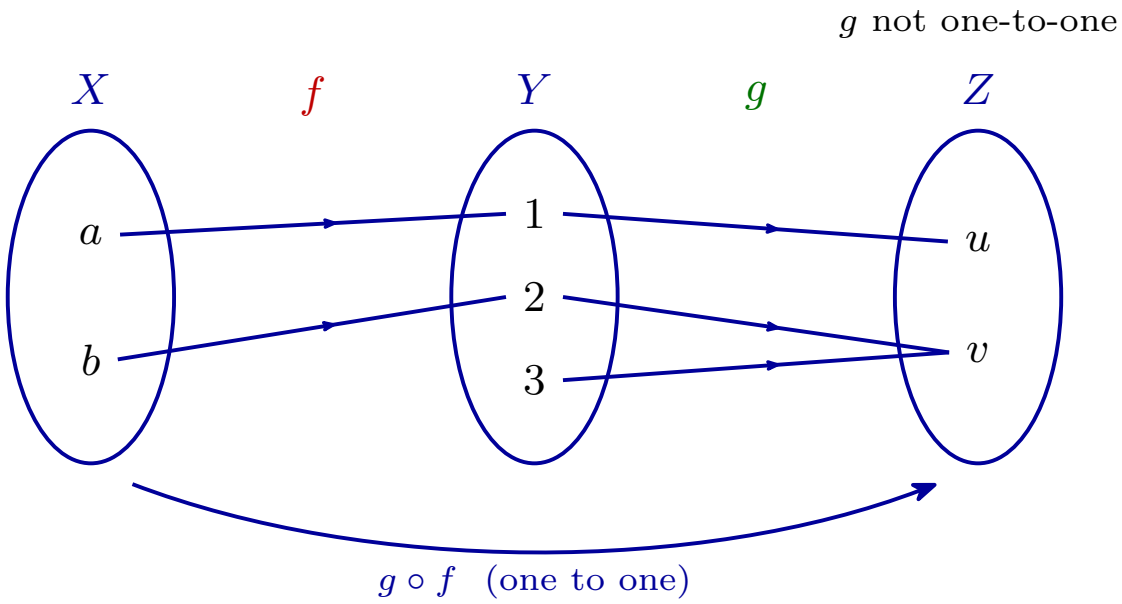
Then g is **not one-to-one** (since $2 \neq 3$ and $g(2) = g(3)$). Now $g \circ f : X \to Z$ is given by $\{(a, u), (b, v)\}$, which **is one-to-one**.

The two counterexamples are mirror images and were built the same way. In (1), f has a gap and g *merges* so the gap does not reach Z. In (4), g merges and f *has a gap* so the merge is never fed. In both, the middle set Y has one more element than it needs — the element that f misses — and that spare element is where the bad behaviour is parked. Small sets, one thing wrong, the wrong thing hidden: that is how to build a counterexample to order.

See also

- composition-of-functions[7] — the definition, the four converse results as theorems, and how to build the counterexamples
- one-to-one-and-onto-functions[8] · inverse-functions[9] — the properties being composed
- proof-techniques[10] — contraposition, which both proofs use
- 2026-09-07-functions-one-to-one-onto-inverse[11] — Lecture 19
- practice-problems[12] — §26, with full worked solutions

[7] courses/math1061/composition-of-functions.pdf
[8] courses/math1061/one-to-one-and-onto-functions.pdf
[9] courses/math1061/inverse-functions.pdf
[10] courses/math1061/proof-techniques.pdf
[11] courses/math1061/2026-09-07-functions-one-to-one-onto-inverse.pdf
[12] courses/math1061/practice-problems.pdf



$$f = \{(a, 1), (b, 2)\} \qquad g = \{(1, u), (2, v), (3, v)\}$$

$$g \circ f = \{(a, u), (b, v)\} \text{ — } \textbf{one-to-one}$$

Figure 5: The counterexample for (4). g merges 2 and 3, so g is not one-to-one — but f never sends anything to 3, so the merge is never exercised: $g \circ f$ sends a and b to different places.