Lecture 19 — Functions: One-to-One, Onto, Inverse

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Pre-work: Video 025 (One-to-one, onto, and inverse functions). The reference notes are one-to-one-and-onto-functions[1] for the two properties and their proof templates, and inverse-functions[2] for bijections, inverses, and the two meanings of f^{-1}; the definition of a function itself is in functions-on-general-sets[3].

> **Date.** The published schedule puts Lecture 19 on Tuesday 08/09, but the course has been running one lecture ahead of it since Lecture 15 (see the note under math1061[4]), which makes this the Monday 07/09 lecture and 2026-09-08-composition-of-functions[5] the Tuesday. Confirm against Blackboard.

[1] courses/math1061/one-to-one-and-onto-functions.pdf
[2] courses/math1061/inverse-functions.pdf
[3] courses/math1061/functions-on-general-sets.pdf
[4] courses/math1061/math1061.pdf
[5] courses/math1061/2026-09-08-composition-of-functions.pdf

Learning goals

- Understanding what a function is, and associated concepts and definitions.
- How to show functions are **one-to-one** (*injective*), **onto** (*surjective*), **bijective**.

The picture throughout is $f : X \to Y$ as two ellipses: X is the *domain*, Y is the *co-domain*, and an arrow f takes x in X to $f(x)$ in Y. An arrow back from $f(x)$ to x would be f^{-1}. The whole topic is when that arrow back is allowed to exist.

The co-domain is part of the function

Defining the co-domain differently can make a function onto.

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 \text{ is not onto} \qquad\qquad g : \mathbb{R} \to \mathbb{R}^{\geq 0},\ g(x) = x^2 \text{ is onto.}$$

The parabola never reaches the negative half of the y-axis, so as a function to $\mathbb{R}$ those values are **missed**. Same rule, same domain, same pairs — only the *declared* co-domain differs, and onto is a statement about the declared co-domain (functions-on-general-sets[6]). Shrink the co-domain to the range and every function becomes onto; that is what g does.

f and g are different functions under the course's definition, even though they contain exactly the same ordered pairs, because a function is written $f : X \to Y$ and Y is part of the declaration.

The inverse function of f does not have the same domain and co-domain as f.

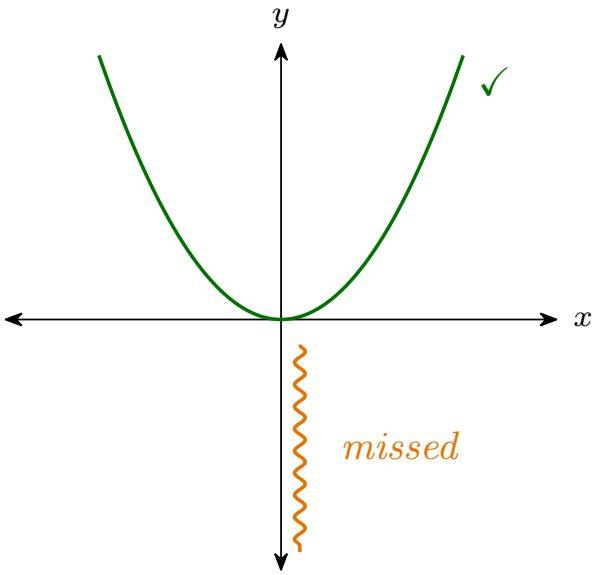
$$f : X \to Y \quad \text{then} \quad f^{-1} : Y \to X.$$

They are swapped. The inverse goes the other way, so its domain is f's co-domain and its co-domain is f's domain.

Definition of a function

> $f : X \to Y$ is $f \subseteq X \times Y$ such that for each $x \in X$, there is a **unique** $y \in Y$ such that $(x, y) \in f$.

[6] courses/math1061/functions-on-general-sets.pdf



$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$: **not onto**

$g : \mathbb{R} \to \mathbb{R}^{\geq 0}$, $g(x) = x^2$: **onto**

the co-domain is declared; shrink it to the range and nothing is missed

Figure 1: The parabola x^2, with the negative y-axis marked *missed* in orange. As a function to $\mathbb{R}$ every negative value is missed, so f is not onto; declare the co-domain to be $\mathbb{R}^{\geq 0}$ and nothing is missed, so g is onto. Same rule, same graph.

The unique y is written $f(x) = y$. There are three things to check: $x \in X$, $y \in Y$, and $\forall x \in X$, $f(x)$ is unique. A fan-out fails the definition: one dot in X with three arrows leaving it is not a function.

Which of the following are functions? The verdicts, each with the reason:

		Verdict	Why
1	$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$	**Function**	
2	$f : \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(n) = n!$	**Function**	$n! = n(n-1)\cdots 3\cdot 2\cdot 1 \in \mathbb{Z}^+$
3	$f : \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(n) = \frac{(n+1)(n+3)}{3}$	**Not a function**	$\frac{(n+1)(n+3)}{3}$ might not be in the co-domain $\mathbb{Z}^+$ — e.g. $n = 1$ gives $\frac{8}{3}$
4	$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$	**Function**	$\{(x, 1) : x \in \mathbb{R}\}$ — constant is fine

		Verdict	Why
5	$f : \mathbb{Q} \to \mathbb{Z}$, $f(\frac{m}{n}) = m$ for all $m, n \in \mathbb{Z}$, $n \neq 0$	**Not a function**	$f(\frac{1}{2}) = 1$ but $f(\frac{2}{4}) = 2$, and $\frac{1}{2} = \frac{2}{4}$ — one input, two values
6	$f : \mathbb{R} \to \mathbb{R}$, $f = \{(x, y) : 3y + 7 = x\}$	**Function**	$y = \dfrac{x - 7}{3}$
7	$f : \mathbb{R} \to \mathbb{R}$, $f = \{(x, y) : y^2 = x\}$	**Not a function**	e.g. $(4, 2), (4, -2) \in f$ — a sideways parabola, which a vertical line meets twice
8	$f : \mathbb{R} \to \mathbb{R}$, $f = \{(x, y) : x = \lvert y \rvert\}$	**Not a function**	e.g. $(1, 1), (1, -1) \in f$ — a sideways V, same test

Each failure breaks one of two clauses. **3** lands outside the co-domain (the $y \in Y$ clause). **5**, **7** and **8** give one input two values (*unique*) — and 5 is the subtle one, because it looks like a rule until you remember that a rational number has many names. A rule that depends on *how the input is written* rather than on *what the input is* is not well defined, which is the standard name for exactly this failure.

7 and 8 are the **vertical line test** from functions-on-general-sets[7], run on curves that are not graphs of y against x — the two red curves in Figure 2.

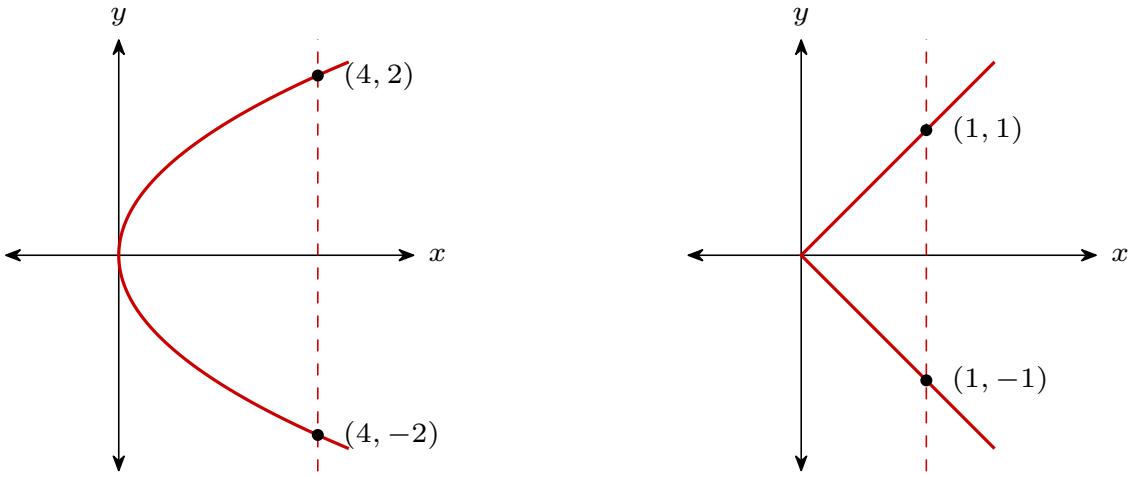
Injective and surjective

Let f be a function, $f : X \to Y$.

> **Injective function (one-to-one function):** $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
>
> **Surjective function (onto function):** $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.

One-to-one is also written *one-one*. The two failures are drawn in Figure 3:

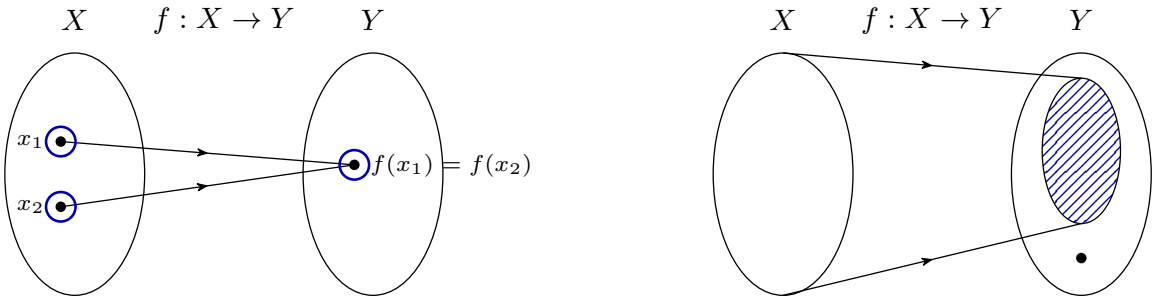
- **not one-to-one** — x_1 and x_2 in X, both arrows landing on the same dot $f(x_1) = f(x_2)$ in Y.
- **not onto** — the whole domain pours into the range, an inner ellipse of Y, and one dot in Y lies outside it: **range** $\subset$ **co-domain**.



$y^2 = x$ — not a function

$x = |y|$ — not a function

Figure 2: Items 7 and 8. $y^2 = x$ is a parabola opening to the right; $x = |y|$ is a V opening to the right. The dashed vertical line meets each curve twice — at $(4, 2)$ and $(4, -2)$, at $(1, 1)$ and $(1, -1)$ — and each pair of points is the witness that the set of pairs is not a function.



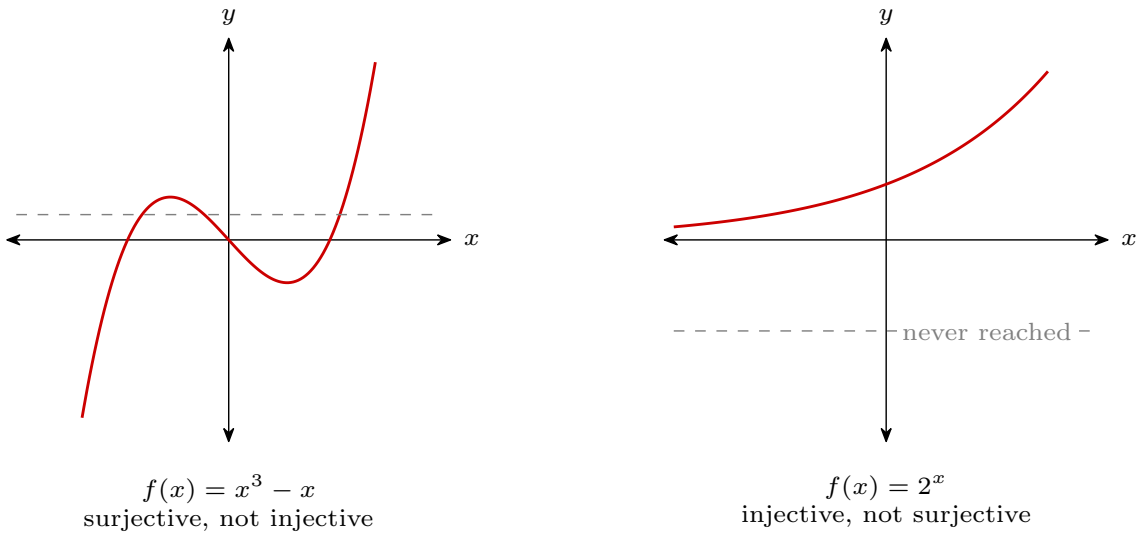
not one-to-one

range ⊂ co-domain
not onto

Figure 3: The two ways a function can fall short. Left: *not one-to-one*, two different inputs sharing one output; the blue rings mark the two sources and the one target. Right: *not onto*, the whole domain funnelled into a range (hatched in blue) that is a proper subset of the co-domain; the dot outside the hatching is a y that no x reaches. Domain on the left, co-domain on the right, names above.

Two examples on the plane, graphed in Figure 4:

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - x \text{ is surjective but not injective.}$$
$$f : \mathbb{R} \to \mathbb{R},\ f(x) = 2^x \text{ is injective but not surjective.}$$



$f(x) = x^3 - x$
surjective, not injective

$f(x) = 2^x$
injective, not surjective

Figure 4: The two graphed examples. The cubic $x^3 - x$ takes every real value, so it is onto, but takes the value 0 three times (and every value in its middle band three times), so it is not one-to-one; the dashed horizontal line is the *horizontal line test*. The exponential 2^x never repeats a value, so it is one-to-one, but never reaches 0 or anything below it, so as a function to $\mathbb{R}$ it is not onto.

The two properties are independent: Figure 4 has one of each, and the constant function $f(x) = 1$ above has neither.

How to show it

There are four templates, two for *showing* a property and two for showing its *failure*:

To show $f : X \to Y$ is …	Do
one-to-one	suppose that $f(x_1) = f(x_2)$ and show that $x_1 = x_2$

[7] courses/math1061/functions-on-general-sets.pdf

To show $f : X \to Y$ is …	Do
not one-to-one	find elements $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$
onto	suppose that $y \in Y$ and find an $x \in X$ such that $f(x) = y$
not onto	find an element $y \in Y$ such that for all $x \in X$, $f(x) \neq y$

The pattern is the same one as for sets (2026-09-03-set-properties-and-functions[8]): a **universal** claim (is one-to-one, is onto) needs a general argument from an arbitrary element, and its **negation** needs one witness. Which is why the two "not" rows are so much shorter to do — and why a proof of one-to-one that checks a few values proves nothing.

Worked: a bijection onto the odd integers

> Consider the function $g : \mathbb{Z} \to \mathbb{Z}^{\text{odd}}$ defined by $g(x) = 2x + 3$. (a) Is g an injective function? (b) Is g a surjective function?

Yes and **Yes**. That is the intuition; the proofs follow.

(a) Proof. Let $x_1, x_2 \in \mathbb{Z}$ such that $g(x_1) = g(x_2)$. Hence

$$2x_1 + 3 = 2x_2 + 3, \qquad 2x_1 = 2x_2, \qquad x_1 = x_2.$$

$\therefore g$ is one-to-one. □

(b) Proof. Let $y \in \mathbb{Z}^{\text{odd}}$. Take $x = \dfrac{y-3}{2}$. Since y is odd, $y - 3$ is even, so $\dfrac{y-3}{2} \in \mathbb{Z}$. And

$$g\left(\frac{y-3}{2}\right) = 2\left(\frac{y-3}{2}\right) + 3 = (y-3) + 3 = y,$$

so $\forall y \in \mathbb{Z}^{\text{odd}}$, $\exists x \in \mathbb{Z}$ s.t. $g(x) = y$. $\therefore g$ onto. □

The thinking behind (b) is *not* part of the proof: **want to find $x \in \mathbb{Z}$ such that $g(x) = y$; $2x + 3 = y$; $x = \frac{y-3}{2}$.** That is scratch work — solving for x backwards — and the proof then runs it forwards: name the candidate, check it is in the domain, check g sends it to y. The "check it is in the domain" line is the one that gets skipped: it is exactly where the oddness of y is used, and without it the argument would "prove" $g : \mathbb{Z} \to \mathbb{Z}$ onto, which is false.

[8] courses/math1061/2026-09-03-set-properties-and-functions.pdf

Worked: a function on pairs

> Define a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f((x, y)) = (y + 1, 2x)$. For example, $f((0, 0)) = (1, 0)$ and $f((-2, 3)) = (4, -4)$.

In the example $f((0, 0)) = (1, 0)$, the input's second entry y feeds the first entry of the output, and the input's first entry x feeds the second. The inputs and outputs are *pairs*, and the coordinates cross over.

(a) Prove that f is one-to-one. Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$ and $f((x_1, y_1)) = f((x_2, y_2))$. Then $(y_1 + 1, 2x_1) = (y_2 + 1, 2x_2)$. So

$$y_1 + 1 = y_2 + 1 \text{ and } 2x_1 = 2x_2, \qquad y_1 = y_2 \text{ and } x_1 = x_2.$$

So $(x_1, y_1) = (x_2, y_2)$. □

The first entries give one equation, the second entries give another, and the **and** between them matters. Two ordered pairs are equal exactly when both coordinates agree, so one equation of pairs is two equations of numbers, and both are needed.

(b) Prove that f is onto. Suppose $(a, b) \in \mathbb{R} \times \mathbb{R}$ *(co-domain)*. Take $\left(\frac{b}{2}, a - 1\right) \in \mathbb{R} \times \mathbb{R}$ *(domain)*, and

$$f\left(\left(\tfrac{b}{2}, a - 1\right)\right) = \left((a - 1) + 1,\ 2\left(\tfrac{b}{2}\right)\right) = (a, b).$$

$\therefore f$ is onto. □

The scratch work behind the candidate: want $f((x, y)) = (a, b)$; $(y + 1, 2x) = (a, b)$; $y + 1 = a$ and $2x = b$; $y = a - 1$, $x = \frac{b}{2}$. Same method as for g above, one coordinate at a time. Here the domain check is trivial — $\frac{b}{2}$ and $a - 1$ are real whenever a and b are — which is why it is written as a bare *(domain)* label rather than argued.

This function returns in 2026-09-08-composition-of-functions[9], where its inverse is computed from exactly this working.

Inverse functions

> A function that is both one-to-one and onto is said to be a **bijection**.
>
> A bijection $f : X \to Y$ has an **inverse function** $f^{-1} : Y \to X$.

$f : X \to Y$ takes $x_1 \mapsto f(x_1)$ and $x_2 \mapsto f(x_2)$, and $f^{-1} : Y \to X$ runs the other way.

The two failure cases are drawn in Figure 5:

- **not one-to-one ⇒ no inverse** (green) — two dots of X going to one dot of Y; the orange arrow back from that dot is marked **?**, because it would have to go to two places.

[9] courses/math1061/2026-09-08-composition-of-functions.pdf

- **not onto ⇒ no inverse** (purple) — a dot of Y outside the range; the arrow back from it is marked **?**, because there is nowhere for it to go.



Figure 5: Why an inverse needs *both* properties. Left (green): if f is not one-to-one, the arrow back from a shared output has two candidates and cannot be a function. Right (purple): if f is not onto, the arrow back from an unreached y has no candidate at all. An inverse function has to send *every* $y \in Y$ to *exactly one* x — onto supplies the *every*, one-to-one supplies the *exactly one*.

Q2 from the pre-class questions

Let $f : X \to Y$ be a bijective function. Which one of the following is true?

		Why
a)	The inverse function of f does not exist.	a bijection is exactly the case where the inverse *does* exist
b)	**For all $y \in Y$, $f^{-1}(y) = x$ where x is the unique element in X such that $f(x) = y$.**	x exists since f is onto; it is unique since f is one-to-one
c)	f^{-1} is a function from X to Y.	the inverse runs $Y \to X$
d)	X and Y are finite.	nothing about bijections needs finiteness — $g(x) = 2x + 3$ above is a bijection between infinite sets

Option b) is the definition of f^{-1}, and its two reasons are the whole topic in one sentence: **onto gives existence, one-to-one gives uniqueness.**

The notation f^{-1} is used in two different ways

This distinction is important.

> **1.** If $f : X \to Y$ is **any** function and A is any subset of Y, then the **inverse image** or **preimage** $f^{-1}(A)$ of A is defined by
>
> $$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$
>
> Notice that the preimage function f^{-1} is a function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$.
>
> **2.** If $f : X \to Y$ is any **bijective** function then the **inverse function of** f is the function $f^{-1} : Y \to X$ where for all $y \in Y$
>
> $$f^{-1}(y) = x \text{ where } x \text{ is the unique } x \in X \text{ such that } f(x) = y.$$
>
> Notice that the inverse function f^{-1} is a function $f^{-1} : Y \to X$.

Both are drawn in inverse-functions[10]. In the first, f is many-to-one and a blob $f^{-1}(A)$ in X collects everything that lands in the blob A in Y. In the second, f is *bijective (one-to-one and onto)* and a single arrow f^{-1} runs back from y to x.

The difference is in what goes in and what comes out:

	Input	Output	Exists for
preimage $f^{-1}(A)$	a **subset** of Y	a **subset** of X	**every** function
inverse function $f^{-1}(y)$	an **element** of Y	an **element** of X	**bijections only**

Preimages are defined in functions-on-general-sets[11] and nothing about them has changed; what is new is that the same symbol now also has a second, much more restricted meaning, and **the braces are what tell you which one is meant**.

[10] courses/math1061/inverse-functions.pdf
[11] courses/math1061/functions-on-general-sets.pdf

Worked: preimage or inverse function?

> Let $f : \{1, 2, 3, 4\} \to \{a, b, c\}$ be the function $f = \{(1, a), (2, a), (3, b), (4, b)\}$. For each of the following statements about the function f, decide whether the statement is true or false.

The arrow diagram is Figure 6: $1, 2 \to a$; $3, 4 \to b$; c untouched.

	Statement		Why
(1)	The inverse function f^{-1} of f does not exist.	**T**	since f is not a bijection
(2)	The preimage of the set $\{a\}$ is $f^{-1}(\{a\}) = \{1, 2\}$.	**T**	$\{a\}$ is a subset of the co-domain; 1 and 2 map to a (no others map to a)
(3)	$f^{-1}(a)$ is not defined.	**T**	a is an element of the co-domain; here the f^{-1} notation represents the inverse function, which is not defined for this f
(4)	$f^{-1}(\{c\}) = \emptyset$.	**T**	$\{c\}$ is a subset of the co-domain; the pre-image of $\{c\}$ is $\emptyset$ since no element maps to c

All four are true, and (2)–(4) are the same lesson three times: **read the braces**. $\{a\}$ and $\{c\}$ are sets, so f^{-1} means preimage and the answer is a set — possibly empty, which is an answer and not an error. a alone is an element, so f^{-1} would have to mean the inverse function, which this f does not have.

See also

- one-to-one-and-onto-functions[12] — the two definitions, the four proof templates, and more worked examples
- inverse-functions[13] — bijections, the inverse function, and the two meanings of f^{-1}

[12] courses/math1061/one-to-one-and-onto-functions.pdf
[13] courses/math1061/inverse-functions.pdf

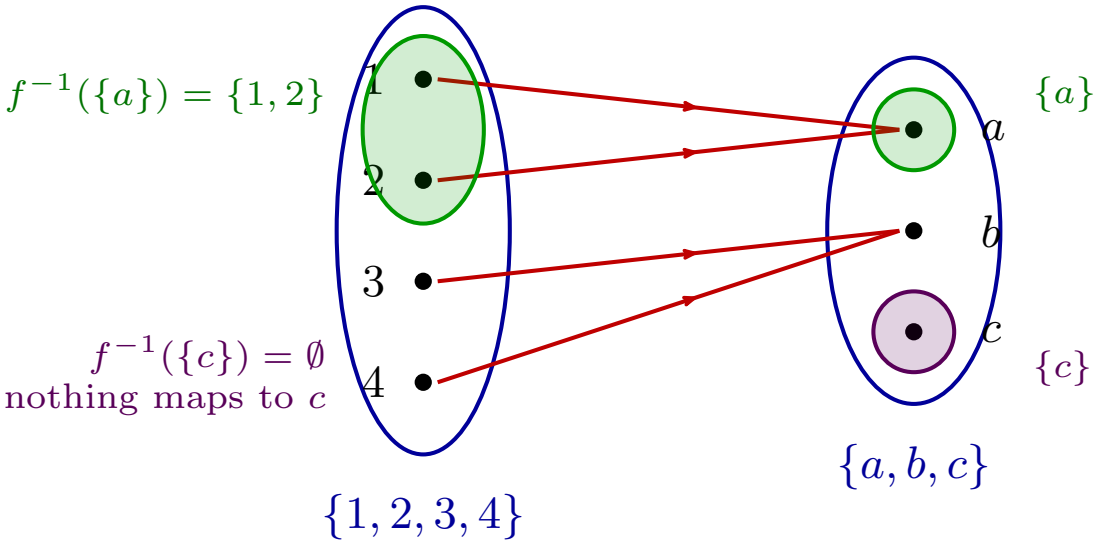


Figure 6: The function f as an arrow diagram. It is neither one-to-one (1 and 2 share a, 3 and 4 share b) nor onto (c is never reached), so there is no inverse function and $f^{-1}(a)$ means nothing. But preimages exist for every function: $f^{-1}(a)$ collects 1 and 2 (green), and $f^{-1}(c)$ collects nothing (purple, empty).

- functions-on-general-sets[14] — the definition of a function, co-domain versus range, image and preimage
- 2026-09-03-set-properties-and-functions[15] — Lecture 18 · 2026-09-08-composition-of-functions[16] — Lecture 20
- practice-problems[17] — §25, with full worked solutions

[14] courses/math1061/functions-on-general-sets.pdf
[15] courses/math1061/2026-09-03-set-properties-and-functions.pdf
[16] courses/math1061/2026-09-08-composition-of-functions.pdf
[17] courses/math1061/practice-problems.pdf