Lecture 17 — Set Theory Definitions and Examples

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Pre-work: Video 021 (Set Theory Definitions) and Video 022 (More definitions and examples of sets). See sets-and-set-notation[1] for the reference material — every definition below, with the worked examples and the diagrams.

Learning goals

1. Understand the distinction between **"is an element of"** $x \in A$ and **"is a subset of"** $A \subseteq B$.

[1] courses/math1061/sets-and-set-notation.pdf

2. Be comfortable with notation for sets including

$$x \in A, \quad A \subseteq B, \quad A \subset B, \quad A \cap B, \quad A \cup B, \quad A - B, \quad A^c, \quad \mathcal{P}(A), \quad A \times B$$

Element, subset, proper subset

What is the difference between "is a subset of" $\subseteq$ and "is a proper subset of" $\subset$? A four-way comparison settles it:

	$\subseteq$	$\subset$
$\mathbb{Z}$ against $\mathbb{R}$	$\mathbb{Z} \subseteq \mathbb{R}$	$\mathbb{Z} \subset \mathbb{R}$
$\mathbb{Z}$ against itself	$\mathbb{Z} \subseteq \mathbb{Z}$	$\mathbb{Z} \not\subset \mathbb{Z}$

A set is always a subset of itself, and never a **proper** subset of itself. That single row is the entire difference — exactly parallel to $\leq$ against $<$.

What is the difference between "is a subset of" and "is an element of"? A concrete universe shows it better than a symbol:

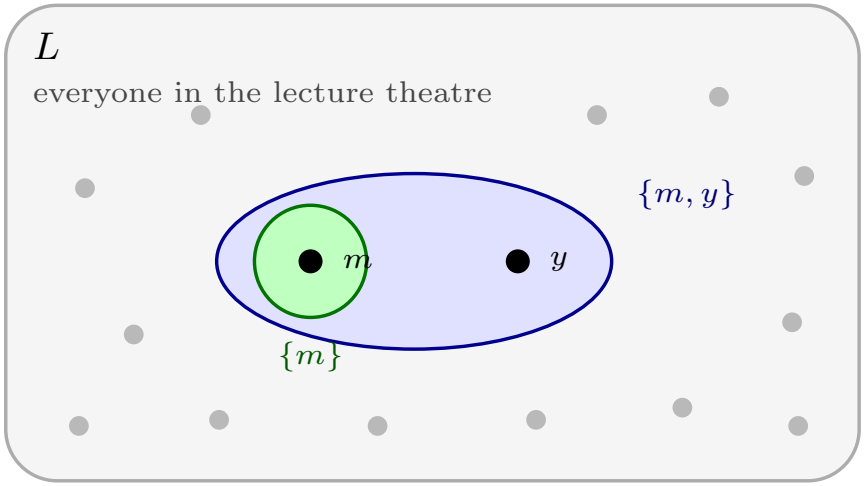
> Let $L :=$ {everyone in the lecture theatre}, and let $m :=$ *me* and $y :=$ *you*.

Four memberships hold: $m \in L$, $y \in L$, $\{m\} \subseteq L$, $\{m, y\} \subseteq L$. The two braced objects are sets. A person is an **element**; a *group* of people is a **subset**. One is a person, the other is a collection of people. Just as $2 \neq \{2\}$ (below), $m \neq \{m\}$ — a person is not a group containing one person.

The other people in the room are drawn into Figure 1 for a reason: $\{m, y\}$ is *a* subset of L, not *the* subset. Every group you can pick out of the room is one, and there are a great many of them — how many exactly is the power set fact further down this page.

Why isn't $\mathbb{Z} \in \mathbb{Q}$ — aren't all integers rational numbers? They are, and that is exactly the claim $\mathbb{Z} \subseteq \mathbb{Q}$. But $\mathbb{Z} \in \mathbb{Q}$ says $\mathbb{Z}$ is *one of the rational numbers*, and $\mathbb{Z}$ is a **set of numbers**, not a number. The instinct is right; the symbol is wrong.

Set difference — do the backslash and the minus sign mean the same thing? Yes. $A - B$ can be written $A \setminus B$. Purely a matter of which textbook you are reading.



elements — a *person*:
$m \in L$ $\quad y \in L$

subsets — a *group* of people:
$\{m\} \subseteq L$ $\quad \{m, y\} \subseteq L$

$m \neq \{m\}$ — a person is not a group of one person

Figure 1: The lecture theatre example drawn out. A person in the room is an element of L; any group of people in the room is a subset of L. The two symbols relate different kinds of thing, which is why m and {m} are not the same object even though the group has exactly one member.

Q2 from the pre-class questions

True or false? $\mathbb{Z} \in \mathbb{Q}$

False. $\mathbb{Z}$ is a set of numbers, so it is not an element of $\mathbb{Q}$; the true statement is $\mathbb{Z} \subseteq \mathbb{Q}$.

Take care with the distinction between "is an element of" and "is a subset of".

And one level down: **a single element is not the same as the set containing a single element**, so $2 \neq \{2\}$.

Worked: is it an element?

Determine whether each of the following is true or false.

	Statement		Reasoning
a)	$4 \in \{4\}$	**True**	$\{4\}$ contains 4 as an element
b)	$4 \in \{\{4\}\}$	**False**	$\{\{4\}\}$ does **not** contain 4 as an element — it does contain $\{4\}$ as an element
c)	$\{4\} \in \{2, 3, 4\}$	**False**	$\{2, 3, 4\}$ does **not** contain $\{4\}$ as an element — it does contain 4 as an element

(b) and (c) are parallel: in each case the set contains *something*, just not the thing being asked about. $\{4\} \subseteq \{2, 3, 4\}$ **is** true — (c) fails on the symbol, not on the containment.

The reliable technique is to write the set's elements out as a list and ask whether the left-hand object is literally one of them. For $\{\{4\}\}$ that list has one entry, the set $\{4\}$, and 4 is not it.

Subsets and proper subsets

A set S is a **subset** of a set T, denoted $S \subseteq T$, if every element that is in S is also in T. Formally we say $\forall x$, if $x \in S$ then $x \in T$.

A set S is a **proper subset** of a set T, denoted $S \subset T$, if $S \subseteq T$ **and** there is an element in T that is not in S.

Examples: $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{Z} \subseteq \mathbb{Z}$; $\mathbb{Z} \subset \mathbb{Q}$ but $\mathbb{Z} \not\subset \mathbb{Z}$.

> For sets S and T, to prove that $S \subseteq T$, you consider an **arbitrary** element of S and show that it is also in T. That is, we suppose $x \in S$ and prove that $x \in T$.

Worked: prove one set is a subset of another

> **(a)** Let $D = \{n \in \mathbb{Z} \mid n \equiv 2 \pmod 8\}$ and let E be the set of even integers. Prove that $D \subseteq E$.

Writing out both sets first makes the shape of the proof obvious:

$$D = \{\dots, 2, 10, 18, \dots\} \qquad E = \{\dots, 2, 4, 6, 8, \dots\} = \{n \in \mathbb{Z} \mid n \equiv 0 \pmod 2\}$$

Rewriting E as a congruence class is the move that makes the two sets comparable — both are now "$\equiv$ something", and the proof is arithmetic.

> *Proof.* Let $x \in D$. Then $x \equiv 2 \pmod 8$, so $\exists k \in \mathbb{Z}$ such that
>
> $$x = 8k + 2 = 2(4k + 1) + 0$$
>
> $\therefore x \equiv 0 \pmod 2$, $\therefore x \in E$. Thus $D \subseteq E$. ■

> **(b)** Is $D \subset E$?

Yes. $4 \in E$ but $4 \notin D$, so $D \subseteq E$ is proper. One witness is all a proper containment needs.

When are two sets equal?

> For sets S and T, saying $S = T$ means $\forall x$, $x \in S$ **if and only if** $x \in T$.

Recall that $S \subseteq T$ means $\forall x$, if $x \in S$ then $x \in T$. Since $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$, it follows that $S = T$ is logically equivalent to $(S \subseteq T) \wedge (T \subseteq S)$.

So to prove two sets S and T are equal you do **two separate proofs**: suppose $x \in S$ and prove $x \in T$ (giving $S \subseteq T$), then suppose $x \in T$ and prove $x \in S$ (giving $T \subseteq S$). Together these show $S = T$.

This is a **consequence**, not a convention — it is the biconditional law of logical-equivalence-laws[2] read through the definition of $\subseteq$.

[2] courses/math1061/logical-equivalence-laws.pdf

The empty set

$$\emptyset = \{\ \}$$

Why is $\emptyset \subseteq A$ for every set A? Because "$\forall x$, if $x \in \emptyset$ then $x \in A$" is true for every set A. The hypothesis is never satisfied, so the conditional is **vacuously true** — the F-hypothesis row of the truth table in conditional-statements[3] is doing all the work.

Take the two halves of the conditional separately: $x \in \emptyset$ is **F** — always false, whatever x is — and $x \in A$ is **F/T**, because it could go either way. A conditional with a false hypothesis is true regardless of the conclusion, so the implication holds for every x and every A.

Do not confuse this with $\emptyset \in A$. There are some sets that do contain the empty set, but there are lots of sets that do not.

Why is $A \subseteq A$ for every set A? Because "$\forall x$, if $x \in A$ then $x \in A$" is true for every set A — this time because the conclusion restates the hypothesis.

> **Lemma.** There is only one empty set.
>
> *Proof (by contradiction).* Suppose that $\emptyset_1$ and $\emptyset_2$ are distinct sets each having no elements, so $\emptyset_1 \neq \emptyset_2$. Since $\emptyset_1$ has no elements, it is a subset of $\emptyset_2$, that is, $\emptyset_1 \subseteq \emptyset_2$. Also since $\emptyset_2$ has no elements, we have $\emptyset_2 \subseteq \emptyset_1$. Thus $\emptyset_1 = \emptyset_2$ by definition of set equality, and we have reached a contradiction. ■

Hence $\emptyset_1 = \emptyset_2 = \emptyset$ — *the* empty set, which is why it gets a symbol of its own.

Unions and intersections

Let $A_1 = \{1, 2, 3\}$, $A_2 = \{1, 2, 4\}$ and $A_3 = \{3, 4, 5\}$.

$$A_1 \cup A_2 = \{1, 2, 3, 4\} \qquad A_1 \cap A_2 = \{1, 2\} \qquad A_1 \cap A_3 = \{3\} \qquad A_2 \cap A_3 = \{4\}$$

$$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5\}$$

$$\bigcap_{i=1}^{3} A_i = (A_1 \cap A_2) \cap A_3 = \{1, 2\} \cap \{3, 4, 5\} = \emptyset$$

All three pairwise intersections are non-empty, and the triple intersection is still empty. Pairwise overlap does not propagate to the whole family.

[3] courses/math1061/conditional-statements.pdf

Worked: an infinite union and an infinite intersection

Determine the following sets, recalling the interval notation $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

$$\bigcup_{i=0}^{\infty} \{i, -i\} = \{0, -0\} \cup \{1, -1\} \cup \{2, -2\} \cup \cdots = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \mathbb{Z}$$

$$\bigcap_{i=0}^{\infty} [-i,\, i] = \{0\}$$

For the intersection, the intervals **grow** with i — $[0, 0] \subseteq [-1, 1] \subseteq [-2, 2] \subseteq \cdots$ — so the intersection is pinned by the smallest, the $i = 0$ term $[0, 0] = \{0\}$. The nested intervals are drawn on a number line in sets-and-set-notation[4].

The power set

Let $S = \{1, 2, 3\}$. Then $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

The elements of a power set are **sets**, which resets the $\in/\subseteq$ trap one level up:

$\{1, 2\} \in \mathcal{P}(S)$	it is one of the eight things listed
$\{1, 2\} \nsubseteq \mathcal{P}(S)$	that would need 1 and 2 to be elements of $\mathcal{P}(S)$
$\{\{1, 2\}\} \subseteq \mathcal{P}(S)$	its one element, $\{1, 2\}$, is in $\mathcal{P}(S)$

Since $\emptyset \subseteq A$ for every set A, it follows that $\emptyset \in \mathcal{P}(A)$ for every set A.

Fact. If T is a **finite** set that contains n elements, then $|\mathcal{P}(T)| = 2^n$.

This fact is **important**. The **finite** qualifier is part of it, and the 2^n comes from building a subset by making n independent in-or-out choices.

[4] courses/math1061/sets-and-set-notation.pdf

Worked: element or subset of a power set?

True or false? $\{1, 3, -1\} \subset \mathcal{P}(\mathbb{Z})$

False, by walking the levels carefully:

$$1, 3, -1 \in \mathbb{Z} \ \therefore \{1, 3, -1\} \subseteq \mathbb{Z} \ \therefore \{1, 3, -1\} \in \mathcal{P}(\mathbb{Z}) \ \text{ so } \ \{1, 3, -1\} \not\subset \mathcal{P}(\mathbb{Z})$$

It is an **element** of the power set, not a subset of it. For it to be a subset, its own elements 1, 3, -1 would have to be subsets of $\mathbb{Z}$ — and they are integers, not sets. The statement is off by exactly one level.

Worked: write down each power set

Number the elements while writing them and then check each answer against 2^n, which is the cheapest way to catch a dropped brace:

Power set	Value	Check
$\mathcal{P}(\{a\})$	$\{\emptyset,\ \{a\}\}$	$\lvert\{a\}\rvert = 1$, so $2^1 = 2$
$\mathcal{P}(\emptyset)$	$\{\emptyset\}$	$\lvert\emptyset\rvert = 0$, so $2^0 = 1$
$\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\})$	$\{\emptyset,\ \{\emptyset\}\}$	$\lvert\{\emptyset\}\rvert = 1$, so $2^1 = 2$
$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\{\emptyset, \{\emptyset\}\})$	$\{\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset, \{\emptyset\}\}\}$	$\lvert\{\emptyset, \{\emptyset\}\}\rvert = 2$, so $2^2 = 4$

The line to get right is $\mathcal{P}(\emptyset) = \{\emptyset\}$: the empty set has exactly one subset — itself — so its power set has **one** element and is **not** empty.

Cartesian products

Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$ and $A_3 = \{6\}$.

$$A_1 \times A_2 = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$$
$$A_1 \times A_2 \times A_3 = \{(1, 3, 6), (1, 4, 6), (1, 5, 6), (2, 3, 6), (2, 4, 6), (2, 5, 6)\}$$

with $|A_1 \times A_2| = |A_1| \cdot |A_2| = 2 \times 3 = 6$. The product is drawn as a grid of points in sets-and-set-notation[5].

Let $S = \{a, b, c\}$ and $T = \{c, d\}$.

[5] courses/math1061/sets-and-set-notation.pdf

- $(d, c) \notin S \times T$ since $d \notin S$ — **order matters**.
- $(c, d) \in S \times T$ since $c \in S$ and $d \in T$.
- $S \times T$ contains $3 \times 2 = 6$ elements.

The familiar case is $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$, often denoted $\mathbb{R}^2$.

Two more facts: if $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$; and for finite sets, $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$.

> **Notation issue to be aware of.** (a, b) can be used for an **ordered pair**. (a, b) can also be used as an **interval**, representing the real numbers x where $a < x < b$. Only context distinguishes them.

Partitions

> A finite or infinite collection of **non-empty** sets $\{A_1, A_2, A_3, \dots\}$ is a **partition** of a set A if and only if
>
> $$A = A_1 \cup A_2 \cup A_3 \cup \cdots \qquad \text{and} \qquad A_i \cap A_j = \emptyset \text{ for each } i, j \geq 1,\ i \neq j$$

Examples. A partition of $\mathbb{Z}$ is given by $\{E, O\}$ where E is the even integers and O the odd integers. Another partition of $\mathbb{Z}$ is $\{\{0\}, \mathbb{Z}^+, \mathbb{Z}^-\}$ — the same set, partitioned two different ways.

Worked: partitions

> **(a)** Let $S = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod 3\}$, let $T = \{x \in \mathbb{Z} \mid x \equiv 2 \pmod 4\}$, and let $R = S \cup T$. Is $\{S, T\}$ a partition of the set R? Explain why or why not.

No. The covering condition $R = S \cup T$ holds by definition, but disjointness fails:

$$10 \equiv 1 \pmod 3 \ \Rightarrow\ 10 \in S \qquad \text{and} \qquad 10 \equiv 2 \pmod 4 \ \Rightarrow\ 10 \in T$$

So $10 \in S \cap T$, hence $S \cap T \neq \emptyset$, and $\{S, T\}$ is **not** a partition of R. Stripe the two congruence classes along a number line — pink for S, green for T — and a tick carrying both colours is a point of $S \cap T$.

The grey capsules mark every tick both stripes cover in the range drawn — 10 is the witness above, and -2 works equally well. Two witnesses are no better than one here: disjointness is a *universal* claim, so a single overlap kills it.

> **(b)** State two different partitions of the set $\mathbb{Z} \times \mathbb{Z}$.

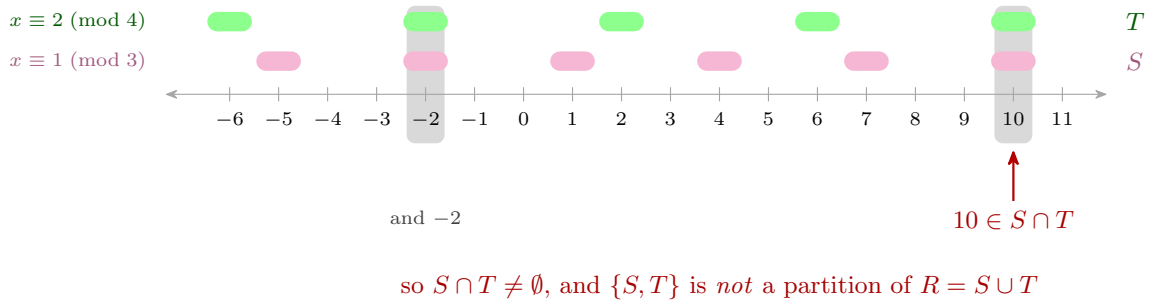


Figure 2: Finding the counterexample: the two congruence classes striped along the integers — S every third integer (pink row), T every fourth (green row) — and read off for a tick that both stripes cover. The grey capsules mark those ticks, which are the overlap.

By the sign of the first coordinate:

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \geq 0\} \qquad L = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a < 0\}$$

$\{L, R\}$ is a partition of $\mathbb{Z} \times \mathbb{Z}$.

By the parity of the second coordinate:

$$E = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \equiv 0 \pmod 2\} \qquad O = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \equiv 1 \pmod 2\}$$

$\{E, O\}$ is another partition of $\mathbb{Z} \times \mathbb{Z}$.

Picture $\mathbb{Z} \times \mathbb{Z}$ as a lattice of dots carrying **both** partitions at once: a block over the right half of the lattice against a block over the left half for $\{L, R\}$, and horizontal stripes alternating row by row for the parity split $\{E, O\}$ — the same "cut the whole thing up and check nothing is covered twice" picture as Figure 2, one dimension up.

The general recipe: **any property that every element either has or lacks splits the set into two parts that partition it**, provided neither part is empty. Made exact later in the course by equivalence relations.

See also

- sets-and-set-notation[6] — the reference note for all of the above
- proving-set-properties[7] · functions-on-general-sets[8] — where Lecture 18 takes this

[6] courses/math1061/sets-and-set-notation.pdf
[7] courses/math1061/proving-set-properties.pdf
[8] courses/math1061/functions-on-general-sets.pdf

- modular-arithmetic[9] — the congruence classes in the subset proof and the partition question
- conditional-statements[10] · logical-equivalence-laws[11] — vacuous truth, and the biconditional behind set equality
- 2026-08-31-solving-recurrence-relations[12] — Lecture 16
- 2026-09-03-set-properties-and-functions[13] — Lecture 18
- practice-problems[14] — §21 Set Theory Definitions, with full worked solutions
- practice-problems[15] — §22 More Definitions and Examples of Sets

[9] courses/math1061/modular-arithmetic.pdf
[10] courses/math1061/conditional-statements.pdf
[11] courses/math1061/logical-equivalence-laws.pdf
[12] courses/math1061/2026-08-31-solving-recurrence-relations.pdf
[13] courses/math1061/2026-09-03-set-properties-and-functions.pdf
[14] courses/math1061/practice-problems.pdf
[15] courses/math1061/practice-problems.pdf