Lecture 7 — Direct Proof, Counterexample

Table of contents

Pre-work: Video 009 (Direct Proofs and Counterexamples). First lecture of the number theory and methods of proof topic — see proof-techniques[1] for the reference definitions, proof templates and the catalogue of common proof errors.

Learning goals

- How to write a direct proof.
- How to disprove a statement by exhibiting a counterexample.

Student questions and comments

What precise definitions should we know for proofs and disproofs? Know the definitions of even, odd, prime, composite, rational, divides — each is an "if and only if", usable in both directions. They're tabulated in proof-techniques[2].

How much detail to give in a proof? Demonstrate facts **using the definitions**.

[1] courses/math1061/proof-techniques.pdf
[2] courses/math1061/proof-techniques.pdf

Two facts used throughout, worth naming when you rely on them:

- **Fact**: the sum, difference, and product of integers is an integer.
- **Fact**: every integer is either even or odd — $\mathbb{Z}$ splits into $\mathbb{Z}^{\text{even}}$ and $\mathbb{Z}^{\text{odd}}$.

Note on "not prime": for $n \in \mathbb{Z}^{>1}$, "n not prime" means "n composite". For $n \in \mathbb{Z}$ generally, "n not prime" means "composite or $n \leq 1$" — the primes and composites together only cover $\mathbb{Z}^{\geq 2}$.

Proof or disproof of $\forall x \in D, P(x) \to Q(x)$

Recall the truth table — the only row that makes $p \to q$ false is p true, q false:

p	q	$p \to q$
T	T	T
T	F	**F**
F	T	T
F	F	T

Direct proof (shows the statement is true):

> Let $x \in D$ and suppose $P(x)$. ... Hence $Q(x)$. $\square$

Disproof by counterexample (shows the statement is false):

> Let $x = \cdots$. Then $x \in D$ and $P(x)$ is true. However, $Q(x)$ is false. So x is a counterexample and the statement is false.

Student comment: why do you suppose that $P(x)$ is true, when $P(x) \to Q(x)$ would still be true if $P(x)$ is false? We just need to rule out the case that $P(x)$ is true and $Q(x)$ is false — so we suppose $P(x)$ is true and prove that in this case $Q(x)$ must also be true. The other rows of the table are true for free.

Q1 from the pre-class questions

Which values for m and n give a counterexample disproving: *for all integers m and n, if $2m+n$ is odd, then m and n are both odd*?

a. $m = 3, n = 5$	m, n both odd — conclusion holds	not a counterexample
b. $m = 3, n = 6$	$2(3) + 6 = 12$ even — hypothesis fails	not a counterexample

c. $m = 4, n = 5$	$2(4) + 5 = 13$ odd, and m is even	**counterexample**
d. $m = 4, n = 6$	$2(4) + 6 = 14$ even — hypothesis fails	not a counterexample

For a counterexample to $p \to q$ we require p **true** and q **false**.

Example proof

$\forall n \in \mathbb{Z}$, if n is odd then $3n + 2$ is odd.

> *Proof.* Let $n \in \mathbb{Z}$ and suppose n is odd. $\leftarrow$ *hypothesis* Since n is odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Now $3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$. Since $k \in \mathbb{Z}$, we have $3k + 2 \in \mathbb{Z}$. Hence $3n + 2$ is odd. $\leftarrow$ *conclusion* $\square$

Note the shape: state the hypothesis, unfold the definition, do the algebra so the target definition is visible, confirm the witness is an integer, then state the conclusion.

Activity 1

For any integer x, if $x + 6 = 4y$ for some integer y, then $\frac{x}{2}$ is an odd integer.

> *Proof.* Let $x \in \mathbb{Z}$ and suppose $x + 6 = 4y$ for some $y \in \mathbb{Z}$. Now $x = 4y - 6$, so $x = 2(2y - 3)$. Hence $\frac{x}{2} = \frac{2(2y-3)}{2} = 2y - 3 = 2(y - 1) - 1$. Thus, because $y \in \mathbb{Z}$, we have $2y - 3 \in \mathbb{Z}$ so $\frac{x}{2} \in \mathbb{Z}$; and $y - 1 \in \mathbb{Z}$, so by definition $\frac{x}{2}$ is odd. $\square$

Both halves are needed: that $\frac{x}{2}$ is an *integer*, and that it has the *odd* form. ($2(y - 1) - 1$ uses the $n = 2k - 1$ form of odd; $2(y - 2) + 1$ works just as well.)

Activity 2 — prove or disprove

(a) There are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

True. Take $m = 1$ and $n = -1$. Then $\frac{1}{m} + \frac{1}{n} = 1 + (-1) = 0 \in \mathbb{Z}$.

(b) $\forall n \in \mathbb{Z}^{\geq 2}$, n is composite or $n + 1$ is composite.

False. Take $n = 2$: then $n = 2$ and $n + 1 = 3$, which are both prime.

(c) For each integer $n \geq 2$, the product of the first n prime numbers minus 1 is prime.

False. (Hint: try $n = 2, 3, 4, \ldots$) Take $n = 4$: $2 \cdot 3 \cdot 5 \cdot 7 - 1 = 209 = 11 \cdot 19$, which is composite.

Note that $n = 2$ and $n = 3$ both work, which is exactly why "arguing from examples" is on the list of common proof errors in proof-techniques[3].

Activity 4 — disprove by proving the negation

Show that $\exists n \in \mathbb{Z}^+$ such that $n^2 + 5n + 6$ is prime is false, by proving the negation.

The statement is existential, so a single counterexample won't do — the negation is universal and must be proved outright:

> Negation: $\forall n \in \mathbb{Z}^+$, $n^2 + 5n + 6$ is composite.
>
> *Proof of the negation.* Let $n \in \mathbb{Z}^+$. Now $n^2 + 5n + 6 = (n+2)(n+3)$. Since $n \geq 1$ we have $n + 2 > 1$ and $n + 3 > 1$. Thus $n^2 + 5n + 6$ is composite. $\square$

Because $n \geq 1$ forces $n^2 + 5n + 6 \geq 2$, "not prime" here really does mean "composite".

See also

- proof-techniques[4] · quantified-statements[5]
- 2026-08-13-proof-by-contradiction[6] — Lecture 8
- practice-problems[7] — §9 Direct Proofs and Counterexamples, with full worked solutions

[3] courses/math1061/proof-techniques.pdf
[4] courses/math1061/proof-techniques.pdf
[5] courses/math1061/quantified-statements.pdf
[6] courses/math1061/2026-08-13-proof-by-contradiction.pdf
[7] courses/math1061/practice-problems.pdf